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Anticausal Analysis of Feedback Amplifiers

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This paper discloses a technique for the direct analysis of linear active circuits, avoiding the solution of simultaneous equations. This is done by representing the circuit in such a way that the signal variables (currents and voltages) are determined sequentially: we only allow a signal variable to depend upon previously determined signal variables, not upon signal variables yet to be determined. Such a circuit is representable by a cascade signal flow graph, a graph containing no feedback loops. Not all circuits can be so represented, of course, but the number which can is expanded by the technique to be described to include most feedback amplifier configurations. This simplification in linear amplifier analysis allows us to trace a clear path from rough design approximations to exact computer analysis. The extension of the analysis to include the effect of nonsaturating nonlinearities is indicated but not developed here.

I. INTRODUCTION

Feedback regulators as human artifacts have been here for a long time. An early one (perhaps the first[1]) was a furnace temperature control invented by Cornelius Drebble (1572–1633) who used it in several versions including an incubator for chickens. The flyball governor may have originated with Huygens[2] in the seventeenth century, and was used for speed control of windmills by Thomas Mead and steam engines by James Watt, both in the early nineteenth century. In the same period, a much more diffuse feedback system was promulgated by Adam Smith in his

Wealth of Nations, which proposed tbat economic self-interest of individuals would automatically assure equilibrium of the economic system, without central control.[1]

Mathematical development of governors began with Maxwell,[3] who determined stability conditions for systems up to the third degree. He "hoped that the subject would obtain the attention of mathematicians." Routh, Lyapunov, Hurwitz, and others responded, extending the stability analysis to systems of higber degree.[4] Still the focus was on stability, that is, preventing the system from being useless. The engineer was pretty much on his own to make the system useful. Minorski was such an engineer. He developed an analysis for the *design* of a ship rudder servo in the early 1920s.[5]

In the same period, Black and Dickieson were working together on amplifiers for carrier transmission of telephone signals. Their *design* problem was to reduce nonlinearities in electronic amplifiers so that the several voice channels would not interfere with one another by modulation. Black's first entry in this area was his invention of feedforward,[6] a technique reinvented by many workers in the 1960s, and inspired, as Black relates, by "an approach to another problem, I don't remember what it was, in a lecture by Steinmetz." Black worked out the invention on the night of the lecture, and he and Dickieson got it working in six hours the next day.[7] The second invention was that of feedback,[8,9] which came out of the first invention in the sense that Black understood that it would do the same job of reducing nonlinearity. An appreciation of stability problems came later. Nyquist, with his paper "Regeneration Theory"[10] (unfortunately titled, according to Black), dealt with stability analysis or preventing oscillations—making sure that Black's invention would work. Dickieson has been quoted as saying about this theory of stability, "At last we knew what we were trying to achieve."[11] Bode later set down the theory of feedback amplifier design, which remains a landmark to this day.[12]

In the middle decades of this century, feedback amplifiers received much attention.[13,14] A recent library search turned up some 750 articles on the subject, indicating that the theory is hard to understand. By making the stability problem the central focus, and in solving it superbly well, Nyquist and Bode relegated the *design* problem to a position of lesser importance. What was the design problem? To reduce modulation products in frequency-division multiplex systems. What was the solution? To maximize the magnitude of the feedback signal, consistent with the stability constraint.

This paper questions the usefulness of feedback as a conceptual tool for design.[15] The physical connection of a portion of the output signal of an amplifier to the input is agreed to be a beneficial measure for many applications. The *analysis* of such a physical structure can be made

without recourse to the concept of feedback by a conceptual leap, one which has already been made by many engineers who design circuits. That leap is to reverse the direction of time, and think in terms of the input signal which is required to produce a given, preassigned output signal. The technique is old, having been applied to passive ladder circuits by many workers, although the origin is unclear.[16] Penfield refers to it as the "Guillemin trick,"[17] but many others of the era used it. Its application to active circuits has been much more limited, consisting of a few papers by the present author,[18] Beddoe,[19] and in control theory, by Rosenbrock.[20]

People working in computer-aided design have already rejected the concept of feedback in favor of general circuit analysis programs that calculate the performance of quite complex circuits by various matrix methods. These programs are most valuable in checking the performance of a circuit after it has been designed and before it is committed to production. They tend to be neutral with respect to circuit concepts, giving mostly correct answers as to how circuits, previously given to them by design engineers, will work. When a circuit doesn't work, the design engineer has difficulty tracing the source of the difficulty from the computer results. The CAD expert, on the other hand, complains that he is not brought into the design process early enough. The design, according to him, has been set in concrete. The problem is sometimes cast in terms of interpersonal relations, but I think that it is structural, in that there is a poor match between the intuitive thought process of the design engineer and the general analysis method of the CAD expert. The design method discussed in this paper should help to resolve this question, since it is at home as much on the computer as it is in the mind (potentially) of the designer.

The focus of this paper is on the *design* problem of feedback amplifiers. Sections II and III are tutorial, because the material is old, and may be unfamiliar to many who might like to understand the rest of the paper. Sections IV and V describe the new theory, and Sections VI and VII are concerned with applying it to familiar problems. Section VIII considers the stability question. While the substance of this paper is theoretical, it was derived from practical design experience with several amplifier configurations, the most recent of which is an operational amplifier with 1-GHz unity-gain bandwidth and 1 volt per nanosecond slew rate, to be reported upon later. The conceptual difficulties were discussed quite thoroughly in an in-hours course taught by the author at Bell Labs.

II. CAUSAL AND ANTICAUSAL ANALYSIS—SINGLE SIGNAL VARIABLES REPRESENTING CAUSE AND EFFECT

Two elementary examples will serve to define what is meant by feedback and its relationship to the choice of independent circuit vari-

ables. In Fig. 1a, a Thevenin source is connected to a load conductance: a first equation is written taking the cause, e_G, as the independent variable, giving rise to a loop gain or return ratio, $-G_L R_G$, and a return difference, $F = 1 + G_L R_G$, as shown in the signal flow graph.[21] A second equation is written taking the effect, v_o, as the independent variable, and the cause, e_G, as tbe dependent variable, in which case no loop gain appears, giving unity return difference. In Fig. 1b, the elementary feedback amplifier circuit of Black's feedback patent is shown with a similar set of causal and anticausal equations, showing again that loop gain does not appear under the anticausal formulation. Clearly, then, return ratio is a property of the mathematical description of the circuits, and not of the circuits themselves.

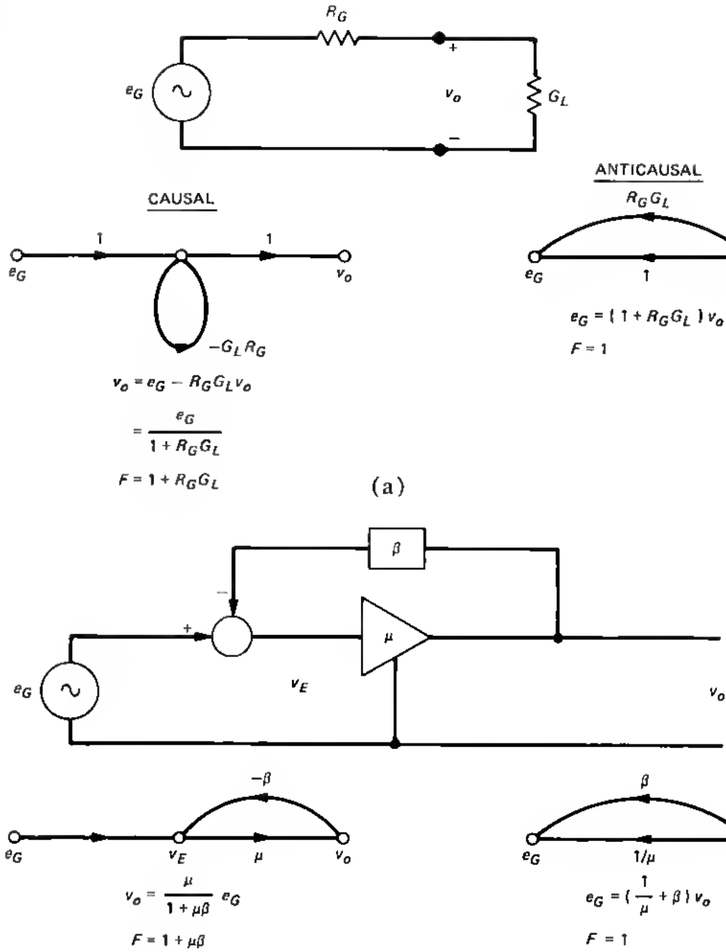
Feedback is seen to be associated with the departure from unity of the denominator in the circuit equation. Since circuit expressions are easier to evaluate (and think about) without denominators, a circuit description which avoids them is conceptually easier to deal with. In general circuit analysis, denominators (or return difference) cannot always be unity, of course, but in many active circuits it will be shown that they can be made to approximate it by appropriate cboice of independent variables.

The word feedback is generally employed in a broader sense than Bode's strict definition of it as return difference. It connotes coupling from output to input of an active circuit, or portion of a circuit, and in this sense can exist, as in Figure 1b with its anticausal equation, without any loop gain. We shall employ the term feedback in this sense even though the description may include no return ratio.

H. Seidel, whose work on feedforward has been of substantial help to the author in clarifying amplifier input-output time relationships, has pointed out that the title of this paper might be interpreted (incorrectly) as describing a physical violation of the principle of causality. No such violation should be inferred. Rather, it is the *analysis* of the causal physical system, proceeding from output to input in a direction from effect to cause, which gives rise to the title of tbis paper.

III. TWO-PORT ANALYSIS USING THE TRANSMISSION MATRIX AND TRANSMISSION MATRIX SIGNAL FLOW GRAPHS

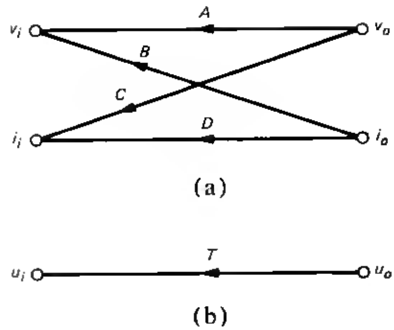
Much useful theory is based on single signal-variable analysis, including some introductory control theory and circuit analysis. For practical circuit work, however, we need to consider at least two signal variables in order to given an adequate, simple description of an amplifier made up from basic parts, such as transistors and passive devices. The simplest of such amplifiers will have an input port and an output port, and we are concerned with the current *and* voltage at each of these ports, four variables in total. Tbe most general way to assign indepen-



(b)

Fig. 1—One-dimensional analysis showing causal and anticausal functional dependencies.

dence and dependence to these four variables, necessary in most cases, is to choose two *independent* variables and two *dependent* variables. There are six possible assignments of port currents and voltages as independent and dependent: one is to choose the port voltages as the set of independent variables. The port currents, then, are the dependent variables, related to the port voltages by an admittance, or y matrix. The choice can profoundly affect the nature of the analysis of the amplifier. In what follows, we use five of the six choices as it suits the occasion, but the basic analysis is involved with the choice of the output current and voltage, the *output signal vector*, $u_o = v_o, i_i$, as the set of independent



$$T = \begin{bmatrix} A & B \\ C & D \end{bmatrix}$$

Fig. 2—Signal flow graph of an amplifier represented by its transmission parameters, and the equivalent transmission matrix signal-flow graph (TMSFG).

variables, and the *input signal vector* $u_i = v_i, i_i$, as the set of dependent variables. The dependent variables are related to the independent variables by the *transmission matrix*, or ABCD matrix:[22,23]

$$\begin{bmatrix} v_i \\ i_i \end{bmatrix} = \begin{bmatrix} A & B \\ C & D \end{bmatrix} \begin{bmatrix} v_o \\ i_o \end{bmatrix} \tag{1}$$

or

$$u_i = Tu_o$$

These equations are shown in signal flow graph form in Fig. 2. In Fig. 2a, eq. (1) is represented by the usual signal flow graph, a graph of directed branches. For each branch, the tail originates at the independent circuit variable, and the nose points toward the dependent variable. The branch value multiplies the value of the independent variable at the tail, and adds the result to the dependent variable value at the nose.

Signal flow graphs are particularly useful in establishing and clarifying functional dependencies in circuits. They are not widely used in circuit analysis and design, however, because of their complexity, even in circuits of quite modest proportions.

In Fig. 2b, a simpler graph, a *transmission matrix signal flow graph* (TMSFG)* connects the output signal vector, u_o, to the input signal vector, u_i, through the matrix branch T. The TMSFG is simply a shorthand way of depicting the graph of Fig. 2a. While signal flow graphs having matrices for the branches were envisioned by Mason[24] and have been studied elsewhere,[25] the application to transmission matrices is new.

* A glossary of terms is given at the end of the paper.

In the transmission matrix signal flow graph, each graph node represents a signal vector consisting of a current and voltage at some point in a circuit, and each branch represents a transmission matrix. The correspondence between the graph and the circuit is direct, with the graph nodes having a direct counterpart in *vector nodes* of the circuit. A *circuit vector node* is defined as a node of the circuit with *only two connections* to it, allowing us to define uniquely the node voltage (to ground) and the node current, which together form the signal vector of the corresponding TMSFG node. The definitions of the transmission parameters are implicit in eq. (1):

$$\begin{aligned}
A &= \frac{\partial v_i}{\partial v_o}, && \text{the reciprocal of } g_{21}\text{, the open circuit voltage gain} \\
B &= \frac{\partial v_i}{\partial i_o}, && \text{the negative reciprocal of } y_{21}\text{, the short circuit transadmittance} \\
C &= \frac{\partial i_i}{\partial v_o}, && \text{the reciprocal of } z_{21}\text{, the open circuit transimpedance} \\
D &= \frac{\partial i_i}{\partial i_o}, && \text{the negative reciprocal of } h_{21}\text{, the short circuit current gain}
\end{aligned} \tag{2}$$

Note that the *ABCD* parameters are all *reciprocals* or negative* reciprocals of familiar forward transfer or gain parameters.

Equation (1) can be written with the transmission matrix taken as a Jacobian matrix, making the equation suitable for analysis of an important class of nonlinear problems:

$$\begin{bmatrix} dv_i \\ di_i \end{bmatrix} = \begin{bmatrix} \dfrac{\partial v_i}{\partial v_o} & \dfrac{\partial v_i}{\partial i_o} \\ \dfrac{\partial i_i}{\partial v_o} & \dfrac{\partial i_i}{\partial i_o} \end{bmatrix} \begin{bmatrix} dv_o \\ di_o \end{bmatrix} \tag{3}$$

The partial derivatives can be expressed as nonlinear functions of the instantaneous output current and voltage, allowing us to find the input voltage and current as nonlinear functions of a preassigned output voltage and current. For a desired sinusoidal output, for example, we can find the input predistortion required to achieve that output. The study of transistor nonlinearities expressed in terms of the partials of eq. (3) is beyond the scope of this paper, and is mentioned here to indicate the

* The parameters which involve i_o are *negative* reciprocals because of differing sign conventions between the *ABCD* parameters, in which the positive direction of current is taken to be outward from the output port, and the h, z, y, and g parameters, in which the reverse is true.

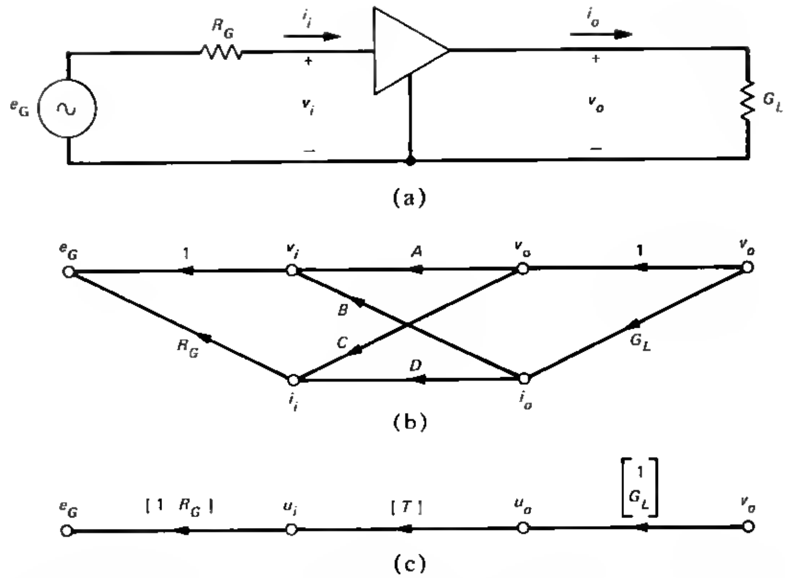


Fig. 3—Connection of a two-port amplifier to source and load, with appropriate signal-flow graphs.

direction of future efforts. For the remainder of this paper, we shall confine our attention to the small-signal case, where the partial derivatives are constants defined at a dc operating point, and are generally functions of frequency.

The calculation of the circuit properties of an amplifier connected as shown in Fig. 3a between a Thevenin source and load conductance is particularly simple if we retain the anticausal direction of analysis that finds the input for a given output. Thus, defining the *loss ratio*, L, as e_G/v_o, we simply add all of the paths from v_o to e_G in Fig. 3b, or, alternatively, perform the matrix multiplication indicated in Fig. 3c. Thus,

$$L = [1 \ R_G] \begin{bmatrix} A & B \\ C & D \end{bmatrix} \begin{bmatrix} 1 \\ G_L \end{bmatrix} \tag{4}$$

$$= A + BG_L + R_G C + R_G D G_L \tag{5}$$

The graphs of Fig. 3 do not include any feedback loops (closed paths). Such a graph is termed a *cascade graph* and has the property that the graph gain (in this case representing the *loss ratio* since the graph source node corresponds to the circuit output) is the sum of all path products from the graph source node to the sink node, from v_o to e_G. With no feedback loops, no denominator appears in the expression for the loss ratio.

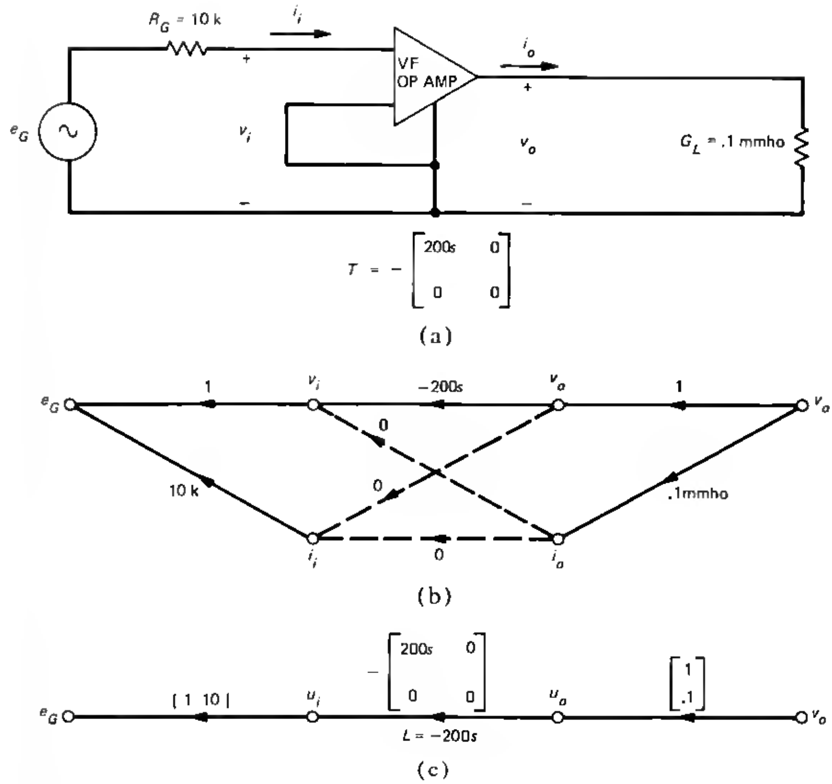


Fig. 4—Loss ratio calculation for a voice-frequency operational amplifier connected between source and load.

Two examples serve to illustrate the loss ratio calculation. In Fig. 4, a voice-frequency operational amplifier is connected between source and load, with the positive input grounded. The transmission matrix of this amplifier can be approximated over most of its frequency range by a single parameter: $A = -200s$, where s is the frequency variable in units of gigaradians per second.* (The low frequency value of input signal is over 100 dB down from the output, and is ignored.) Thus, the loss ratio is $-200s$, as shown in Fig. 4, and is seen not to depend upon the source or load immittances within the range of accuracy of the simple model.

In the signal flow graph of Fig. 4b, the zero value elements in the active path (operational amplifier) transmission matrix are represented by dotted lines; when these are ignored, only a single path exists between the v_o and e_G nodes. The TMSFG is shown in Fig. 4c.

* To save writing powers of 10, we shall adopt the following system of units throughout: the volt, milliampere, and nanosecond are taken as our fundamental units, leading to the derived units of kohms, mmhos, microhenries, picofarads, gigaradians per second (Gr/s), and GHz.

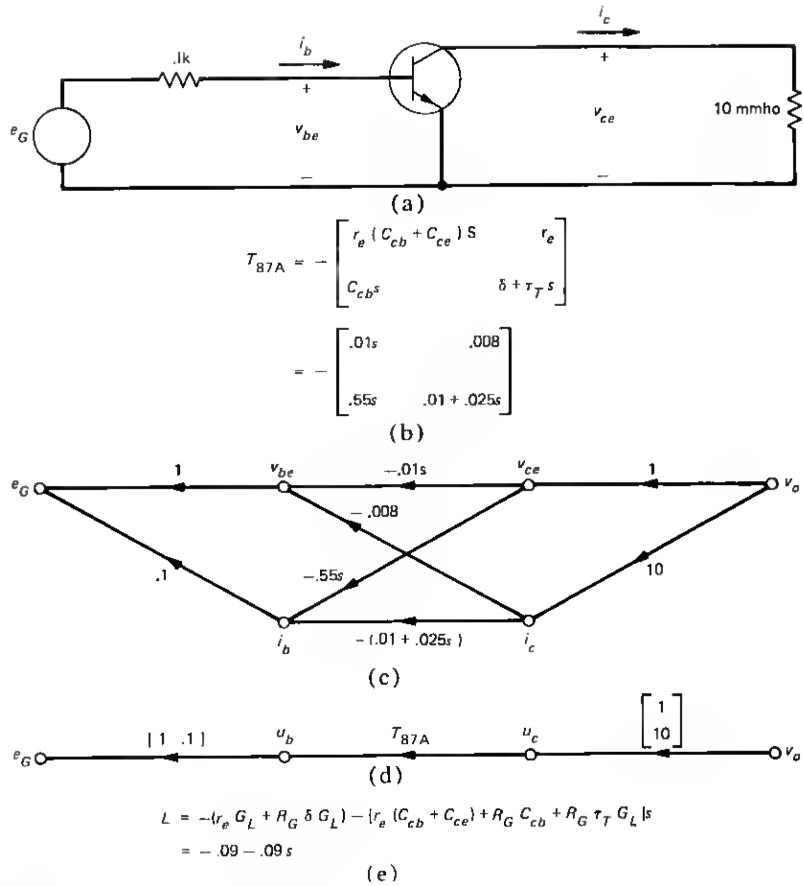


Fig. 5—Loss ratio calculation for a common emitter stage employing a type 87A transistor.

The second example, shown in Fig. 5, is a common emitter transistor stage using the Western Electric type 87A transistor. Accurate characterization of the transmission parameters of this transistor is under way; for purposes of the illustration, we approximate the transmission matrix as shown in Fig. 5b, accurate in magnitude to 1 GHz but somewhat deficient in phase. The transistor parameters are determined at a collector current of 5 mA, and a collector-to-emitter voltage of 3 volts, and are

r_e = emitter resistance	0.008 kohm
C_{CB} = collector-to-base capacitance	0.55 pF
$\delta = 1/h_{fe}$, the reciprocal current gain	0.01
$\tau_T = 1/2\pi f_T$, the current gain time constant	0.025 ns
C_{ce} = collector-to-emitter capacitance	0.7 pF

These parameter values yield the transmission matrix shown in Fig. 5b, whose elements are entered on the signal flow graph of Fig. 5c. The TMSFG is shown in Fig. 5d. The calculation of loss ratio, shown in Fig. 5e, can be made by adding all path products of the signal flow graph, or by performing the matrix operations indicated in the TMSFG. The loss ratio is seen to be a binomial in the frequency variable, with a cutoff frequency of 1 Gr/s, or 0.16 GHz.

This second example points to an advantage of the anticausal approach in determining circuit sensitivities. With feedback loops absent, the input signal is simply the sum of all paths from output to input of the signal flow graph, so that the sensitivity of the loss ratio to r_e, for example, is simply the proportion of the input contributed by r_e. At low frequencies, the r_e contribution is 0.08 out of a total input of 0.09, so that the sensitivity to r_e is 0.08/0.09, or 0.89.

The advantages of the anticausal approach for the simple circuits studied so far are implicit in the removal of feedback loops and therefore denominators from the transmission expressions. It remains to be shown that a method may be developed for retaining these advantages in more complicated circuits with more than mere ladder or cascade coupling.

IV. "FEEDBACK": THE EFFECT OF SPANNING NETWORKS

We define a *spanning network* as a two-port network which is connected between a pair of nonadjacent circuit vector nodes of a cascaded network. In the circuit of Fig. 6a, for example, the conductance G will be considered to be a spanning network around the transistor, and in Fig. 6b, the upper transistor will be considered to be a spanning network around the lower transistor. The choice at tbis point is arbitrary as to which is the spanning network and which is the cascade network. The consideration of active spanning networks is beyond our scope here, but in the case of the circuit of Fig. 6a, the reason for the choice will become clear. The conductance can be represented by its two-port dependent generator equivalent circuit as shown in the figure. Four separate effects are introduced by G, corresponding to the four elements of its y-parameter matrix:

$$y = \begin{bmatrix} G & -G \\ -G & G \end{bmatrix} \tag{6}$$

Clearly, y_{11} and y_{22} load the input and output circuits by shunt conductances equal to G. The generator $y_{12}v_o = -Gv_o$ augments the input *current* by an amount proportional to the output voltage upon which it depends. This y_{12} augmentation is usually the reason for connecting G to the circuit, and the other three effects (of y_{11}, y_{22}, and y_{21}) are side effects, usually deleterious. The fourth effect, introduced by the gen-

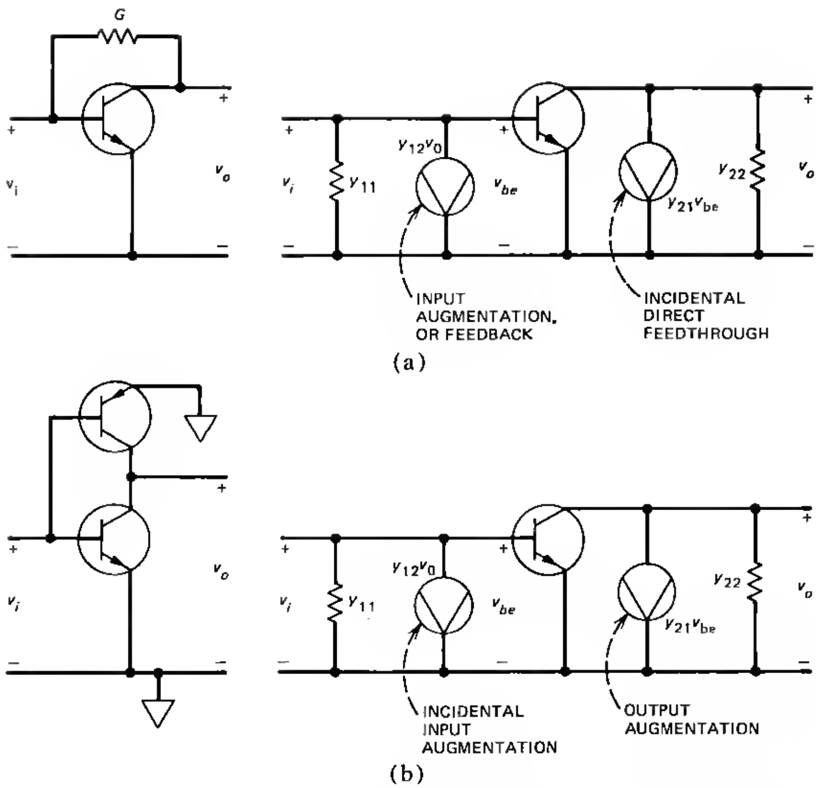


Fig. 6—Two types of spanning networks: (a) input signal augmentation, or feedback type, and (b) output signal augmentation, or feedforward type.

erator $y_{21}v_i$, is *direct feedthrough.* Where the transistor has high gain, e.g., where $v_i \ll v_o$, this effect is incidental, and can often be neglected.

The method of treating a spanning network in anticausal circuit analysis is (*i*) to *represent* its two-port characteristics by one of the four sets of network parameters whose dependent generator equivalent circuits and signal flow graphs are shown in Fig. 7, and (*ii*) to *decompose* its four network parameters into four separate transmission matrices, corresponding to the four effects of input and output circuit loading, input augmentation (or "feedback"), and output augmentation (direct feedthrough).

The four two-port representations of Fig. 7 correspond to the four well-known feedback configurations. The h parameters are chosen to represent a spanning network which provides series-input/parallel-output feedback, the main effect of which is to augment transmission parameter A by h_{12} of the spanning network. We shall term tbis A

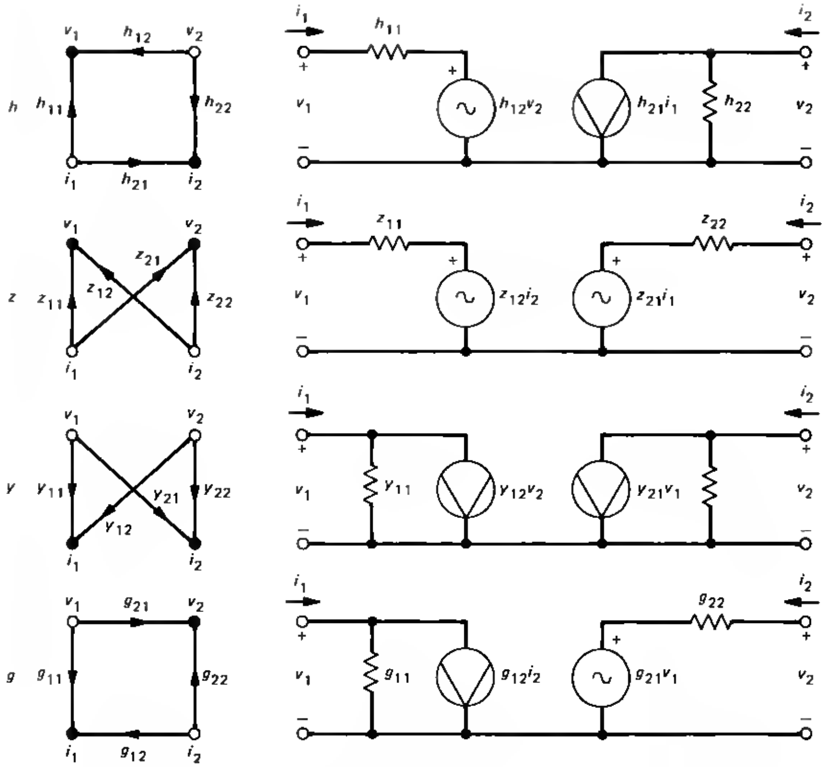


Fig. 7—Signal-flow graphs and equivalent circuits for two-ports corresponding to the h, z, y, and g parameter representations.

feedback. The z parameters augment B by z_{12}, and are appropriate for series-input/series-output, or B feedback. Similarly, y_{12} augments C, providing C feedback, and g_{12} augments D, giving D feedback. We shall consider all four types of feedback in the following section. In this section, we shall consider C feedback (parallel-input/parallel-output feedback) in detail.

A signal-flow graph for the circuit of Fig. 6a is shown in Fig. 8a. The four branches labeled A, B, C, and D represent the transmission parameters of the transistor. The other four (nonunity) branches represent the effect of the four y parameters of the spanning network. Three of these latter branches, corresponding to y_{11} and y_{22}, the input and output loading by the spanning network, and y_{21}, the direct feedthrough to the output through the spanning network, are shown as dashed lines to indicate their lesser importance.

In Fig. 8b, a TMSFG for this circuit is shown. The active path transmission matrix, T_a, includes the four transistor transmission parameters of Fig. 8a. The four branches of Fig. 8a which represent the spanning

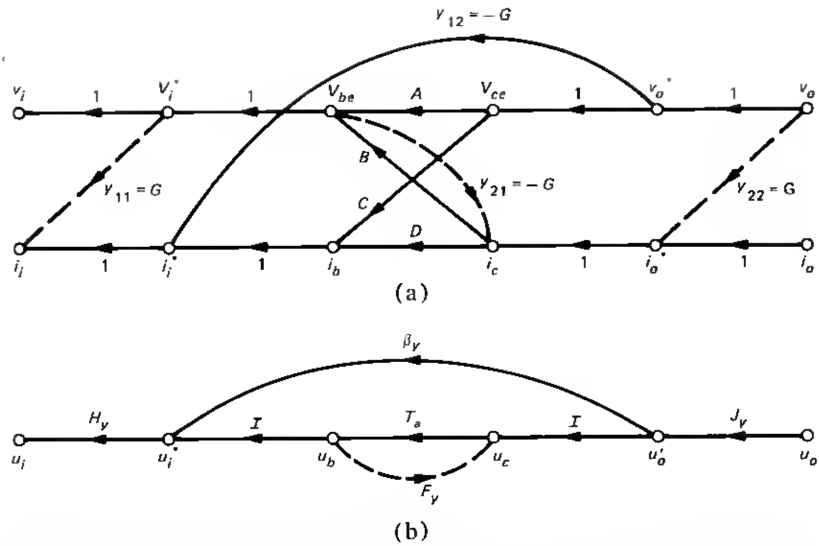


Fig. 8—Development of the signal-flow graph and the TMSFG for the circuit of Fig. 6a. (a) Signal-flow graph, with direct feedthrough and input and output feedback loading branches shown with dashed lines. (b) TMSFG.

network y parameters yield *four separate transmission matrices*, β_y, F_y, H_y, and J_y, defined as follows: β_y is the input augmentation or feedback matrix, given by

$$\beta_y = \begin{bmatrix} 0 & 0 \\ y_{12} & 0 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ -G & 0 \end{bmatrix} \tag{7}$$

F_y is the direct feedthrough, or feedforward matrix, given by

$$F_y = \begin{bmatrix} 0 & 0 \\ y_{21} & 0 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ -G & 0 \end{bmatrix} \tag{8}$$

H_y is the input loading matrix, given by

$$H_y = \begin{bmatrix} 1 & 0 \\ y_{11} & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ G & 1 \end{bmatrix} \tag{9}$$

J_y is the output loading matrix, given by

$$J_y = \begin{bmatrix} 1 & 0 \\ y_{22} & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ G & 1 \end{bmatrix} \tag{10}$$

In these four equations, the matrices containing G were obtained by substituting the y parameters of eq. (6) into the general expressions.

The transmission matrix for the C-feedback amplifier of Fig. 6a can be obtained by evaluating the graph gain of the TMSFG. As shown in Appendix A, this transmission matrix is

$$T = H_y[\beta_y + T_a(I - F_yT_a)^{-1}]J_y \tag{11}$$

The matrix F_yT_a will be called the *return ratio matrix,* and $(I - F_yT_a)^{-1}$ will be termed the *return difference matrix inverse.* These two matrices arise from the presence of a feedback loop in the signal flow graph and in the TMSFG, one which arises from the incidental direct feedthrough, or feedforward from input to output through the spanning network. Where the y_{21} branch can be ignored, F_y can be considered a null matrix, and the graphs become cascade graphs. The transmission matrix of eq. (11) can also be written

$$T = \beta_y + H_yT_a(I - F_yT_a)^{-1}J_y \tag{12}$$

since

$$H_y\beta_yJ_y = \beta_y$$

This equation states that the transmission matrix of the amplifier is the sum of the β_y matrix and the matrix of the active path, which itself is the transmission matrix of the transistor, modified by input and output loading and direct feedthrough. Our next step is to calculate the effect of these modifications of the active path transmission matrix.

To evaluate the effect of direct feedthrough, we begin by finding the return ratio matrix:

$$F_yT_a = \begin{bmatrix} 0 & 0 \\ y_{21}A & y_{21}B \end{bmatrix} \tag{13}$$

whereupon the return difference matrix inverse becomes

$$(I - F_yT_a)^{-1} = \frac{1}{1 - y_{21}B}\begin{bmatrix} 1 - y_{21}B & 0 \\ y_{21}A & 1 \end{bmatrix} \tag{14}$$

The effect of direct feedthrough is a small modification of the transmission parameters of the active path. Thus,

$$T_a' = T_a(I - F_yT_a)^{-1} = \frac{1}{1 - y_{21}B}\begin{bmatrix} A & B \\ C + y_{21}\Delta^t & D \end{bmatrix} \tag{15}$$

where $\Delta^t = AD - BC$ is the determinant of the transmission matrix of the transistor. Ordinarily, $|y_{21}B| \ll 1$ and $|y_{21}\Delta^t| \ll C$, so that the active path remains essentially unaffected by the direct feedthrough.

The loss ratio between a Thevenin source and a load conductance is found as in eq. 4:

$$L = [1 \quad R_G]T\begin{bmatrix} 1 \\ G_L \end{bmatrix} \tag{16}$$

Using eq. (12) for T, and substituting the matrix element values of eqs.

(7) to (10), we obtain

$$L = R_G y_{12} + [1 \quad R_G] \begin{bmatrix} 1 & 0 \\ y_{11} & 1 \end{bmatrix} T_a' \begin{bmatrix} 1 & 0 \\ y_{22} & 1 \end{bmatrix} \begin{bmatrix} 1 \\ G_L \end{bmatrix} \tag{17}$$

$$= R_G y_{12} + [1 + R_G y_{11} \quad R_G] T_a' \begin{bmatrix} 1 \\ G_L + y_{22} \end{bmatrix} \tag{18}$$

The term $1 + R_G y_{11}$ is a potentiometer term arising from the voltage divider action between the source resistance, R_G, and the spanning network input loading admittance, y_{11}, and the term $G_L + y_{22}$ represents the total output load admittance, including the spanning network output loading. Letting $P_G = 1 + R_G y_{11}$ and $G_L' = G_L + y_{22}$, we can write

$$L = R_G y_{12} + \frac{1}{1 - y_{21}B} [P_G A + P_G B G_L' + R_G C - R_G y_{21} \Delta^t + R_G D G_L'] \tag{19}$$

We can recapitulate the above development by identifying each term of this equation with the relevant spanning network effect, comparing it with eq. (5) for the loss ratio without the spanning network. The first term is the input augmentation, or "feedback," which is of course absent from eq. (2–5). This is the reciprocal of the familiar R_F/R_G gain approximation for this circuit, with $R_F = 1/G$. The remaining terms are divided by the return difference, $1 - y_{21}B$, which is ordinarily close to unity. The first term in the brackets, $P_G A$, is the same as that of eq. (5), except that it is magnified by the input loading factor, $P_G = 1 + y_{11}R_G$. The second term is magnified by this term as well as by the increased output loading provided by y_{22}. The third term is unchanged from eq (5). The fourth term is new: ordinarily very small, it constitutes a reverberation of the signal back and forth through the circuit. The fifth term in the brackets is the D term of eq. (5) magnified by the increased load conductance.

The main difference between the two equations is the feedback term, $R_G G$. The loading has a lesser effect but it is not normally negligible. The direct feedthrough effect is normally negligible.

As examples of the use of the above equations, consider the circuits of Figs. 4 and 5 with feedback conductances connected between input and output, as shown in Fig. 9. The loss ratio for the operational amplifier circuit consists of only two terms of eq. (19), since we have assumed that B, C, and D are zero. Hence, Δ^t is also zero, and

$$L = -R_G G + P_G A \tag{20}$$

Since $P_G = 1.1$ and $A = -200s$, we have

$$L = -(.1 + 220s) \tag{21}$$

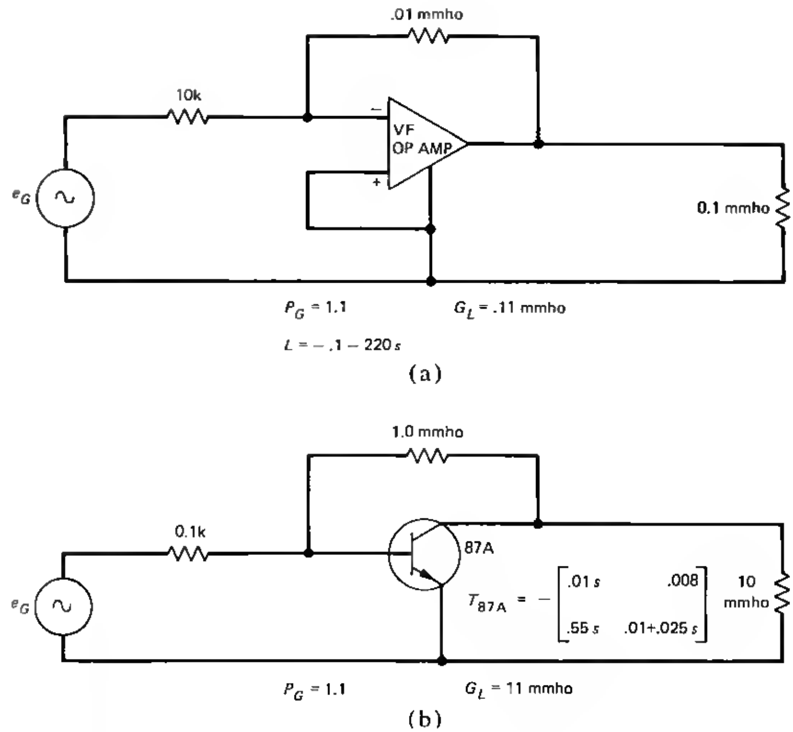


Fig. 9—Examples of loss ratio calculation with spanning network.

Direct feedthrough and output loading are of no concern by our assumption that B and D of the active path are zero. Feedback, or input augmentation, gives the low-frequency value of .1, and the value is A magnified by the input P_G term.

A more substantial example is provided in Fig. 9b, in which the common emitter stage of Fig. 5 is modified by connecting a 1 mmho conductance from input to output. With $R_G = .1$ k, we have $P_G = 1.1$, and with $G_L = 10$ mmho, we have $G_L' = 11$ mmho. From eq. (19), the loss ratio of the transistor stage is

$$L_{87\text{A}} = -.1 - \frac{1}{1 - 0.008}[.011s + .0968$$

$$+ .055s + .00043s + .011 + .0275s] \quad (22)$$

in which the terms are in the order given in eq. (19). The value of Δ^t is taken as $.0043s$, ignoring the s^2 coefficient, since it affects the result only at frequencies higher than the range of approximation of the transistor model (1 GHz). Thus,

$$L_{87\text{A}} = -.209 - .0947s \quad (23)$$

When the direct feedthrough is ignored, the denominator in eq. (22) becomes unity, and the $R_G G \Delta^t$ term (.00043s) drops out, giving

$$L_{87\mathrm{A}} \approx -.208 - .0935s \tag{24}$$

an approximation of which a circuit designer can be proud. When input and output loading are ignored, P_G becomes unity and G_L' reverts to G_L, 10 mmho. This approximation is rougher, but still valuable for circuit thinking:

$$L_{87\mathrm{A}} \approx -.19 - .090s \tag{25}$$

A still rougher approximation is obtained by ignoring certain of the less important transistor parameters (for this case), such as δ and C_{ce}. With these assumptions, the transmission matrix of the transistor becomes

$$T_{87\mathrm{A}} \approx -\begin{bmatrix} .0044s & .008 \\ .55s & .025s \end{bmatrix} \tag{26}$$

and the loss ratio becomes

$$L_{87\mathrm{A}} = -(.18 + .084s) \tag{27}$$

which is roughly 10 percent below the true value. A much rougher approximation is obtained by ignoring the contribution of the active path entirely, a good strategy where the loss ratio is controlled primarily by the feedback. For the case of the 87A, we would obtain $L_{87\mathrm{A}} = .1$, a poor approximation, since r_e contributes to the low frequency loss ratio, and C_{cb} and τ_T provide most of the high-frequency loss. In the case of the op amp, this strategy accurately predicts the low-frequency loss ratio, but obviously cannot account for the increase in loss ratio at high frequencies. The significance of ignoring the active path contribution is that it defines the transmission matrix of the active path as the null matrix. This provides us with a convenient *reference condition* for a feedback circuit. Where Bode defined a reference condition for a feedback circuit as the circuit in which the "tube [active path] is dead," we stand the definition on its bead, and take our reference condition as one in which the active path is very much alive—an ideal two-port amplifier—to be discussed in the next section. The approximation is widely used in operational amplifier applications such as active filter design.

The analysis of the C-feedback amplifier in this section shows that the essential character of the simple anticausal analysis of the circuits of Section II is retained when the y-parameter spanning network is added to the circuit. The cascade nature of the signal-flow graph is essentially retained because tbe loop gain of the inevitable feedback loops is below unity, and for usual feedback circuits, negligible. The sensitivities to circuit elements are easily evaluated. The low-frequency sensitivity of loss ratio to r_e in the 87A feedback circuit, for example, is seen

from eq. (22) to be .0976/.209 = .47, compared with the previously calculated value of .89 for the stage without the spanning network. The contribution of r_e to the input voltage bas not been reduced—it actually has increased slightly because of the output loading by the spanning network and by tbe input P_G term—but the total generator voltage has been increased by the input signal augmentation of the spanning network, tending to swamp out the effect of r_e.

In this approach to active circuit analysis, the functional dependencies have been chosen in such a way that the increase in bandwidth and reduction in sensitivities usually ascribed to feedback are accounted for without the presence of denominators associated with feedback.

V. NOTES ON FEEDBACK THEORY

Equation (12) for the C-feedback stage neatly separates four essentials of a feedback amplifier. The two terms of the equation separate the feedback or spanning network and the active paths. The feedback network matrix, β_y, contains one nonzero element which augments the current at the amplifier input in proportion to the output voltage, as we have seen. The active path consists of four matrices, including T_a, the transmission matrix of the active patb, H and J, the matrices representing the circuit loading by the spanning network at the amplifier input and output respectively, and $(I - F_yT_a)^{-1}$, the return difference matrix inverse representing direct feedthrough. This equation permits a clear definition of the β-matrix; by setting T_a equal to zero, that is, making T_a the null matrix, the second term in the brackets drops out, so that the transmission matrix of the amplifier becomes β_y. We shall define the *reference condition* for the amplifier by setting $T_a = [0]$. Thus, β_y is the transmission matrix of a C-feedback amplifier whose active path has been set in the reference condition. Later, this definition will be extended to A-, B-, and D-feedback amplifiers.

The concept of an amplifier whose input voltage and current are zero for all finite output signal vectors is a serviceable one which is fairly widely used in making rough calculations of gain of feedback circuits. Calling such an amplifier an ideal two-port amplifier* we can state the following.

Theorem: *An ideal two-port amplifier is an amplifier whose transmission matrix is the null matrix.*

Proof: From equation (5), we have

$$L = A + BG_L + R_GC + R_GDG_L = 0 \tag{28}$$

since $e_g/v_o = 0$ by definition. Since the terminations are arbitrary and

* To distinguish it from an ideal operational amplifier, which is a three-port.

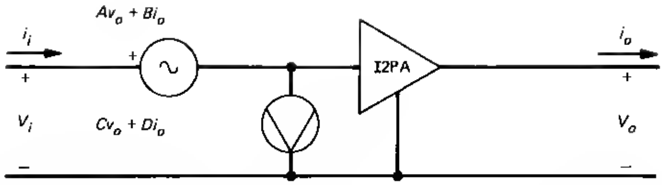


Fig. 10—Dependent generator equivalent circuit for a two-port represented by its transmission parameters.

nonzero, A, B, C, and D must be zero individually. Note that the input and output impedances are indeterminate, since

$$Z_{\text{in}} = \frac{A + BG_L}{C + DG_L} \tag{29}$$

and

$$Z_o = \frac{B + DR_G}{A + CR_G} \tag{30}$$

Impedances will be determined solely by externally applied spanning networks. An ideal two-port amplifier (I2PA) is a circuit element having no parameters to specify it (much like the nullator and norator),[26] and represents a limiting value for an active two-port. It is often useful in drawing equivalent circuits and in modeling; it can, for example, allow us to draw a dependent generator equivalent circuit for a two-port described by the transmission parameters, as shown in Fig. 10.

We can apply the concept of the I2PA to investigate the properties of various feedback configurations. With the active patb of a feedback amplifier set in the reference condition, the resulting transmission matrix is simply the β matrix, without the complicating effect of a nonideal active path. In Fig. 11, the circuits of four *unitary feedback amplifiers* and their associated transmission matrices are shown. A *unitary feedback amplifier* is defined as one whose β matrix has but one nonzero element. Figures 11a and d employ permutative feedback—feedback obtained when the active device leads are permuted.[27] Wben the active path consists of a transistor, these are the common collector and common base stages, respectively. The transmission matrices shown are obtained by inspection, bearing in mind that the input current and voltage of the I2PA are zero. The circuits of Figures 11b and c are duals, with the transmission matrices likewise obtained by inspection.

We can obtain a good approximation to the actual transmission matrix of each of the four circuits of Fig. 11 with a nonideal active path by simply adding the transmission matrix of the active path, with due attention to the sign change introduced in Fig. 11a and d by the permutation of the device leads. This amounts to approximating the transmission matrix

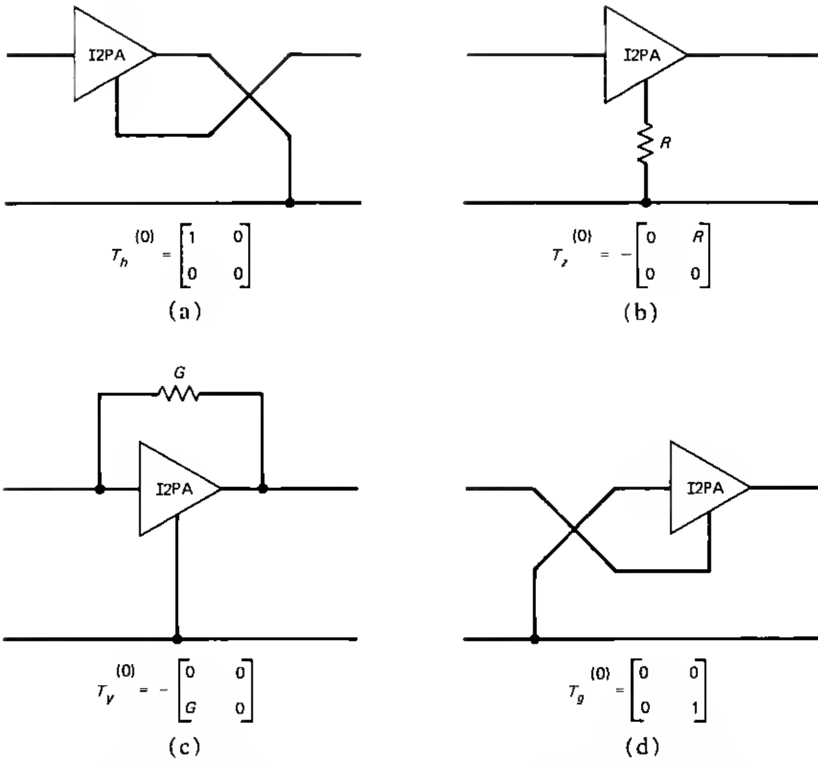


Fig. 11—Transmission matrices of four unitary feedback amplifiers whose active path is in the reference condition. (a) Common collector stage (example of A feedback); (b) emitter resistor feedback (example of B feedback); (c) collector-to-base feedback (example of C feedback); (d) common base stage (example of D feedback).

by the equation

$$T \approx \beta + T_a \tag{31}$$

In the case of the common collector stage, this amounts to approximating the transmission matrix as

$$T_{cc} \approx \begin{bmatrix} 1 - A & -B \\ -C & -D \end{bmatrix} \tag{32}$$

Since permutative feedback is lossless, there are no input and output loading terms, so that the approximation involves ignoring the direct feedthrough. The exact transmission matrix for this stage is derived in Appendix B, and is

$$T_{cc} = \frac{1}{1 - D} \begin{bmatrix} \theta & -B \\ -C & -D \end{bmatrix} \tag{33}$$

where $\theta = 1 - A - D + \Delta^t$, a quantity close to unity which will recur below. Where $D \ll 1$, that is, well below f_T, the approximation of eq. (32) is quite close.

In the case of B-feedback, for which the z-parameter description of the spanning network is appropriate, the approximation of eq. (31) gives

$$T_z \approx \begin{bmatrix} A & B - R \\ C & D \end{bmatrix} \tag{34}$$

The exact value, derived in Appendix C, is

$$T_z = \frac{1}{1 + CR} \begin{bmatrix} A + CR & B - R\theta \\ C & D + CR \end{bmatrix} \tag{35}$$

In comparing this expression with that of eq. (34) we note that the denominator and the multiplication of R by θ are due to direct feedthrough; the CR term added to A comes from input loading by the spanning network, and the CR term added to D comes from output loading by the spanning network. These modifications are small, but may become important at high frequencies, since C represents the (negative) admittance of the collector capacitance, a determining factor in high-frequency performance.

The transmission matrices for the four circuits of Fig. 11 with nonideal active paths are given in Table I. These expressions are intended for computer implementation, since they are complex, and their complexity arises from relatively small corrections on the approximations discussed here. The approximations can be used in the design process.

Table I also includes the transmission matrix of one nonunitary feedback amplifier, a *hybrid feedback amplifier,* incorporating both B- and C-feedback. As can be seen from the table, the matrix for the reference condition includes nonzero elements in all four positions of the matrix. The matrix was obtained by using the transmission matrix elements of T_z as a set of active path elements for the computation of T_y.

As noted above, a spanning network is represented by one of the four parameter sets of Fig. 7. Any one of these parameter sets contains four parameters, each of which generates a transmission matrix; these have been termed β, F, H, and J matrices corresponding to the four effects generated by the spanning network; input augmentation, direct feedthrough, and input and output loading, respectively. Each of the four types of unitary feedback can be represented by the same TMSFG, shown at the top of Table II. The rows of Table II define these four transmission matrices for each of the spanning network parameter sets of Fig. 7. Signs are a problem, since the sign conventions for two-port parameters are different for the parameter sets of Fig. 7 and for the

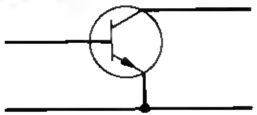
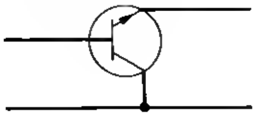
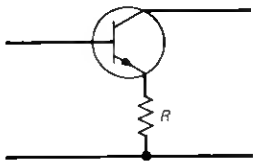
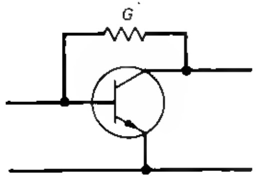
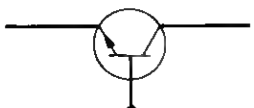
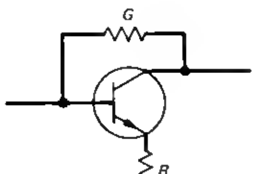
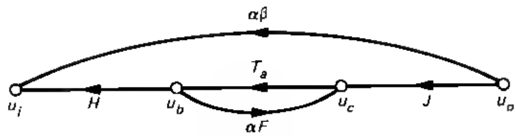
Circuit	Transmission matrix	Reference condition
	$T_a = \begin{bmatrix} A & B \\ C & D \end{bmatrix}$	$\begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$
	$T_{cc} = \frac{1}{1-D}\begin{bmatrix} \theta & -B \\ -C & -D \end{bmatrix}$	$\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$
	$T_z = \frac{1}{1+CR}\begin{bmatrix} A+CR & B-R\theta \\ C & D+CR \end{bmatrix}$	$\begin{bmatrix} 0 & R \\ 0 & 0 \end{bmatrix}$
	$T_y = \frac{1}{1+BG}\begin{bmatrix} A+BG & B \\ C-G\theta & D+BG \end{bmatrix}$	$\begin{bmatrix} 0 & 0 \\ G & 0 \end{bmatrix}$
	$T_{cb} = \frac{1}{1-A}\begin{bmatrix} -A & -B \\ -C & \theta \end{bmatrix}$	$\begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$
	$T_{z/y} = \frac{1}{1+CR+BG-RG\theta}\begin{bmatrix} A+CR+BG-RG\theta & B-R\theta \\ C-G\theta & D+CR+BG-RG\theta \end{bmatrix}$ $\theta = 1 - A - D + \Delta^t$	$\frac{1}{1-RG}\begin{bmatrix} RG & R \\ G & RG \end{bmatrix}$

Table II — Matrix element values for the four types of unitary feedback



TMSFG

Type of feedback	α	β	F	H	J
A	-1	$\begin{bmatrix} h_{12} & 0 \\ 0 & 0 \end{bmatrix}$	$\begin{bmatrix} 0 & 0 \\ 0 & h_{21} \end{bmatrix}$	$\begin{bmatrix} 1 & h_{11} \\ 0 & 1 \end{bmatrix}$	$\begin{bmatrix} 1 & 0 \\ h_{22} & 1 \end{bmatrix}$
B	-1	$\begin{bmatrix} 0 & z_{12} \\ 0 & 0 \end{bmatrix}$	$\begin{bmatrix} 0 & z_{21} \\ 0 & 0 \end{bmatrix}$	$\begin{bmatrix} 1 & z_{11} \\ 0 & 1 \end{bmatrix}$	$\begin{bmatrix} 1 & z_{22} \\ 0 & 1 \end{bmatrix}$
C	1	$\begin{bmatrix} 0 & 0 \\ y_{12} & 0 \end{bmatrix}$	$\begin{bmatrix} 0 & 0 \\ y_{21} & 0 \end{bmatrix}$	$\begin{bmatrix} 1 & 0 \\ y_{11} & 1 \end{bmatrix}$	$\begin{bmatrix} 1 & 0 \\ y_{22} & 1 \end{bmatrix}$
D	1	$\begin{bmatrix} 0 & 0 \\ 0 & g_{12} \end{bmatrix}$	$\begin{bmatrix} g_{21} & 0 \\ 0 & 0 \end{bmatrix}$	$\begin{bmatrix} 1 & 0 \\ g_{11} & 1 \end{bmatrix}$	$\begin{bmatrix} 1 & g_{22} \\ 0 & 1 \end{bmatrix}$

transmission parameters. This is accounted for in Table II by introducing the parameter α, which is -1 for the h and z parameter sets and $+1$ for the y and g parameter sets. The signs of the parameter values are all consistent with conventional practice.

The cascode stage of Fig. 12a illustrates a situation in which the common-base stage effectively removes or reduces certain active parameters of the common emitter stage. The transmission matrix of the cascode stage is

$$T_{\text{cascode}} = T_{a1}(\beta_D - T_{a2})$$

$$= T_{a1}\beta_D - T_{a1}T_{a2} \tag{36}$$

$$= \begin{bmatrix} 0 & B_1 \\ 0 & D_1 \end{bmatrix} - T_{a1}T_{a2} \tag{37}$$

The first matrix of eq. (37) is the transmission matrix of the common emitter stage with A_1 and C_1 removed, the primary effect of which is to remove C_1, the (negative) susceptance of the Miller capacitance. The second term is the negative of the cascaded pair of transistors in the common emitter configuration, a matrix whose elements are much smaller than those for a single stage up to frequencies at which the common emitter gain becomes small.

The process of sorting out the unique character of amplifier configurations is helpful for circuit design. Consider the cascades of unitary

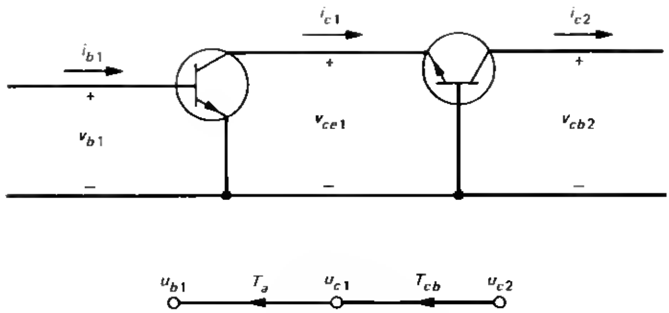


Fig. 12—Cascode stage.

feedback amplifiers of Fig. 13.[28] In Fig. 13a, the cascade of a B feedback amplifier with a C feedback amplifier has a transmission matrix given by T_zT_y of Table I. When both transistors are placed in the reference condition, we observe that the elements of tbe β matrix of the combination are all zero except for the factor RG in the A position, so that the combination is itself a unitary A-feedback amplifier. Reversing the order of the stages, in Fig. 13b, gives a D-feedback amplifier. If we cascade two C-feedback stages (or two B-feedback stages) we find that the β matrix of the combination is null. In Fig. 13c, we note that the C feedback of the second stage augments the input current of that stage, but not the voltage. Since the β network of the first stage senses the input voltage of the second stage, which is zero in the reference condition, no overall feedback arises. The overall loss ratio increases as a result of the input augmentation of the individual stages; the increased input current of the second stage increases the contribution of B_1 and D_1 to the loss ratio, and the feedback around the first stage increases the effect of A_2 and B_2, but the β matrix for the combination is null.

VI. EQUIVALENT LADDER CIRCUITS FOR FEEDBACK AMPLIFIERS

The circuit of any amplifier whose two-port characteristics are sought may be drawn as an *equivalent ladder circuit,* that is, a cascade of active and passive network elements, by the direct expedient of representing circuit couplings among nonadjacent nodes of the ladder by one or more of the dependent generator equivalent circuits of Fig. 7. This will be illustrated by deriving an equivalent ladder circuit for the A-feedback pair of Fig. 14, in which the output voltage of the second stage is divided down in a resistive divider and applied to the emitter of the first stage, where it augments the amplifier input voltage. Since the β network augments A of the active path transmission matrix, the β network is properly represented by its h parameters. The relationship of the h parameters to circuit elements R_E and R_F is given in Fig. 14b, which also defines a

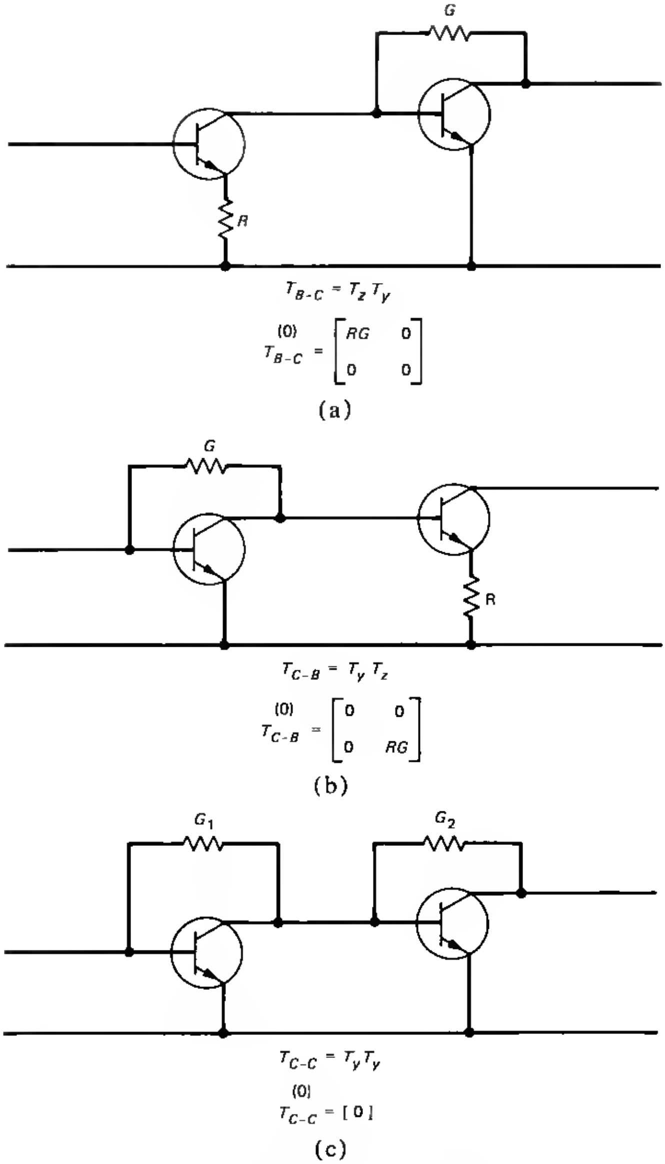


Fig. 13—Cascades of unitary feedback amplifier.

more convenient set of feedback parameters, R_A, G_A, and n_A, where R_A is the parallel combination of R_E and R_F, G_A is the conductance of the series combination of R_E and R_F, and n_A is $R_E/(R_E + R_F)$, which can be considered the turns ratio of the ideal transformer in the network shown. The application of the spanning network is shown in Fig. 14c. The

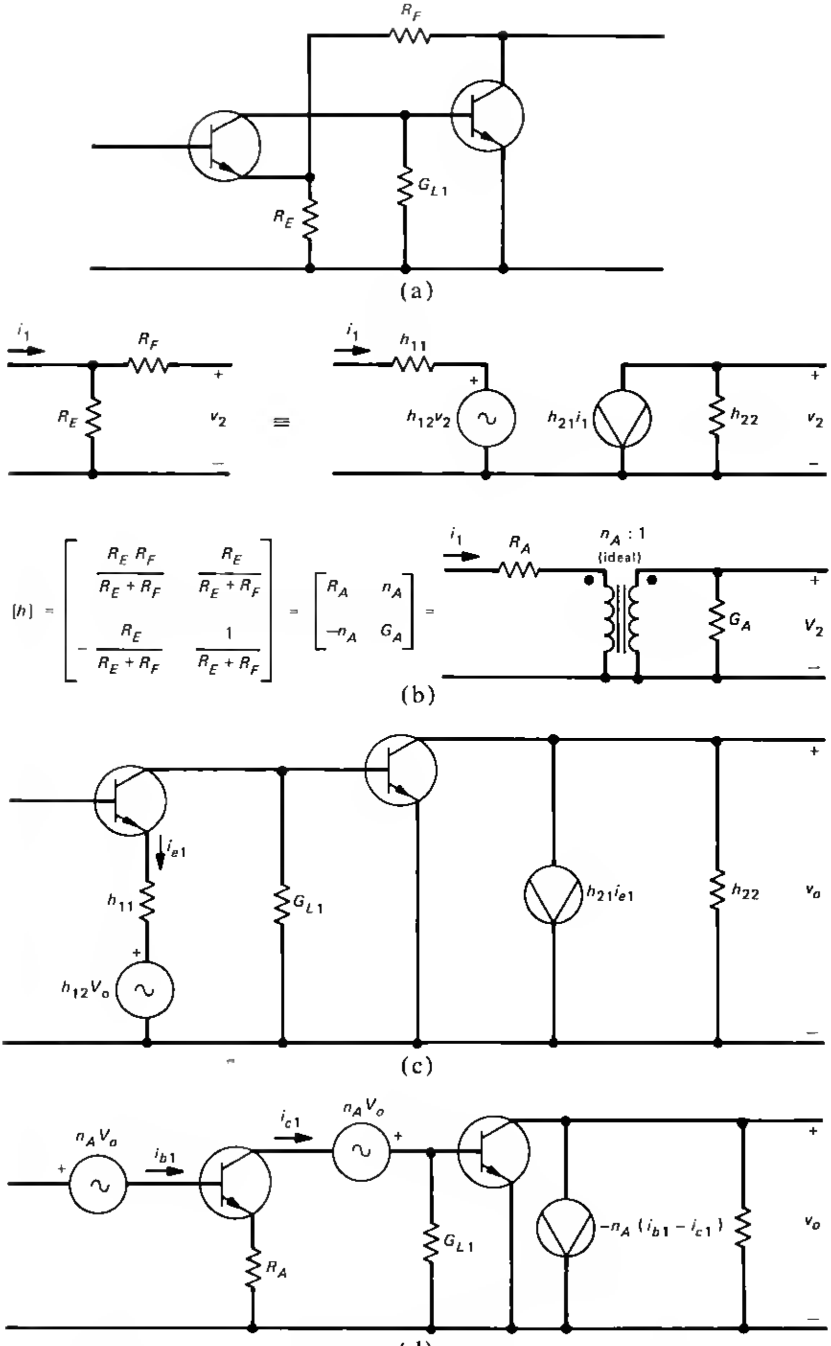


Fig. 14—The A-feedback pair. (a) Circuit; (b) analysis of spanning network; (c) application of dependent generator equivalent circuit of spanning network; (d) equivalent ladder network.

final equivalent circuit of Fig. 14d removes tbe $n_A v_o$ gererator from the emitter circuit of the first stage, replacing it by two generators, one in series with the base lead of the first stage, and one in series with the collector lead. This completes tbe transformation to the ladder configuration, except that we have left R_A in the emitter of the first stage. The reduction to a ladder of elementary active and passive devices would strictly require that tbe local B feedback of tbe first stage be represerted by a separate z-parameter spanning network. To save work, we shall take the single-stage circuits of Table I as elementary building blocks, so that the first-stage active patb will be represented as T_z. Thus, there is no need to reanalyze the single-stage circuits of Table I each time they arise. In computer evaluation, the properties of T_z are derived from T_a in a subroutine. We note that wben the transistors of the circuit of Fig. 14 are placed in the reference condition, the transmission matrix is

$$\beta_A = \begin{bmatrix} n_A & 0 \\ 0 & 0 \end{bmatrix} \tag{38}$$

so that the A-feedback pair is a unitary feedback amplifier.

The ladder equivalent circuit for the D-feedback amplifier of Fig. 15a is derived in an exactly analogous manner, and is shown in Fig. 15b. In this case, the g parameters of Fig. 7 are the appropriate set, since g_{12} relates the input current to the output current. When the transistors are placed in the reference condition, the transmission matrix of the circuit is

$$\beta_D = \begin{bmatrix} 0 & 0 \\ 0 & -n_D \end{bmatrix} \tag{39}$$

so that this, too, is a unitary feedback amplifier.

Simultaneous application of A- and D-feedback is shown in Fig. 15c, and the ladder equivalent circuit is shown in d. The circuit is an extension of the two unitary feedback circuits from which it is derived. When the transistors of this circuit are placed in the reference condition, the expression for the β matrix is complicated. It can be simplified by separating out the effects of G_D and G_A, which we would normally associate with the source and load immittances, respectively. The remaining matrix may be written by inspection, and is the middle matrix of:

$$\beta_{A/D} = \begin{bmatrix} 1 & 0 \\ G_D & 1 \end{bmatrix} \begin{bmatrix} n_A & R_A G_{L1} R_D \\ 0 & n_D \end{bmatrix} \begin{bmatrix} 1 & 0 \\ G_A & 1 \end{bmatrix} \tag{40}$$

If G_{L1} is eliminated (by bootstrapping or by use of an active current source to provide dc for the first stage), the β matrix consists essentially of the two ratios, n_A to establish the voltage gain and n_D to establish the current gain.

This configuration is another instance of a *hybrid feedback amplifier*

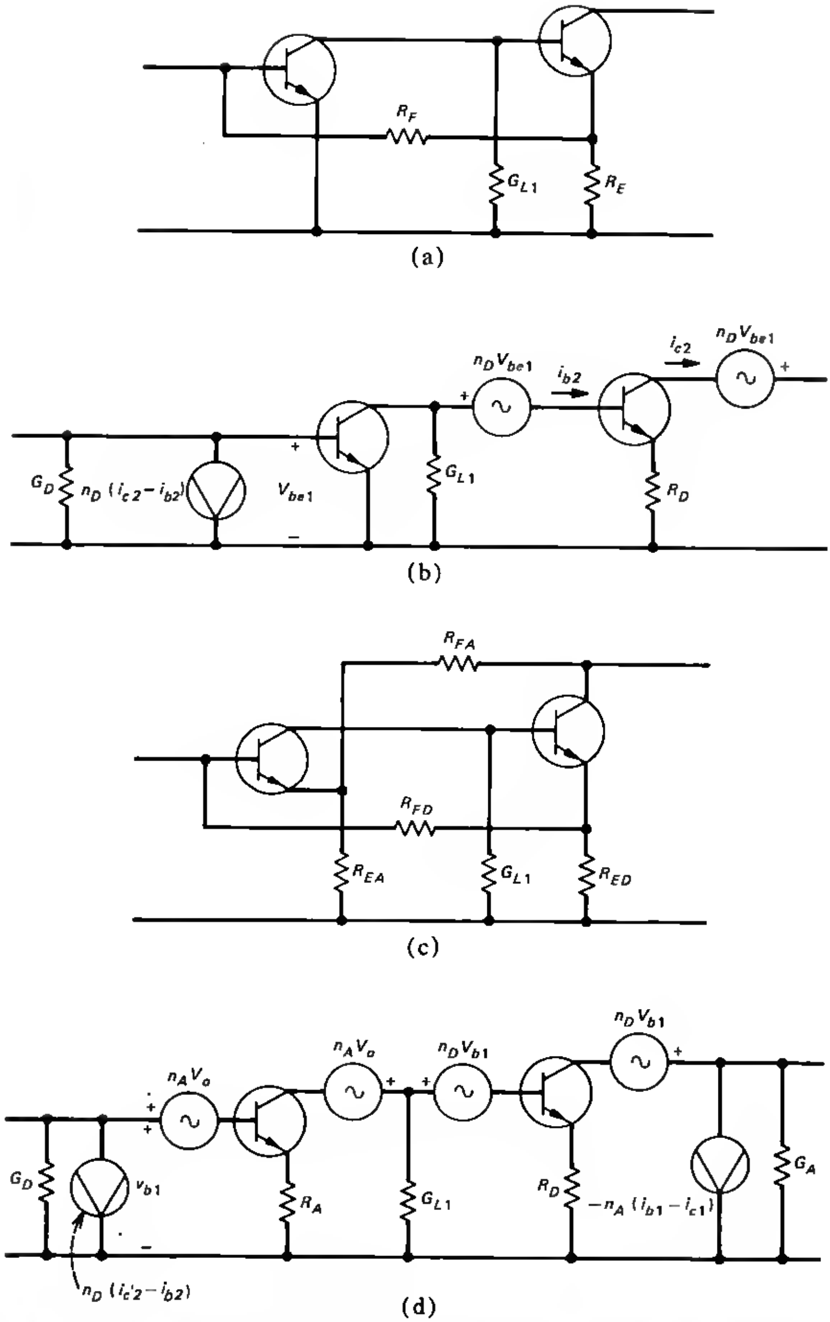


Fig. 15—Further development of equivalent ladder circuits. (a) A D-feedback amplifier and (b) its equivalent ladder circuit. (c) A hybrid A/D-feedback amplifier and (d) its equivalent circuit.

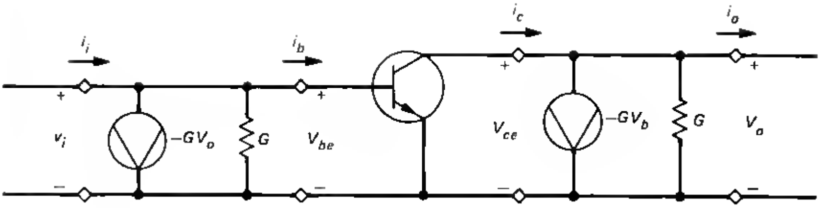
(the first was encountered as the last entry of Table I). Hybrid feedback can be used to provide a desired input and/or output impedance without incurring the power loss associated with build-out resistance or conductance.[29] At the amplifier output, such a build-out incurs a loss of power output capability, while at the input it increases noise. In the present instance, the two generators establish both the input current and voltage, and therefore the input impedance, and contribute only an incidental amount of noise (associated with the input loading of the spanning networks, which could be eliminated by making the spanning networks lossless, by use of transformers rather than voltage dividers). This leads to the surprising thermodynamic conclusion pointed out by Nyquist that such an amplifier cools down the source, since the source pumps noise power into tbis noiseless resistance, and receives no noise power in return.

At the beginning of this section, we stated that any amplifier for which the two-port cbaracteristics are sought may be represented by a ladder network with (shunt) current generators and (series) voltage generators which are dependent upon voltages or currents at nonadjacent circuit nodes. Where parallel active paths are involved, a choice must be made as to which of the two paths is to be taken as the spanning network represented by its h, z, y, or g parameters. Where an active spanning network is involved, such as in feedforward circuits, it is advantageous to assign the role of spanning network in such a way that the loop gain arising from feedforward or direct feedthrough is minimized, since this most closely realizes a cascade graph representation. This procedure is beyond the scope of this paper, but will be treated in a subsequent publication.

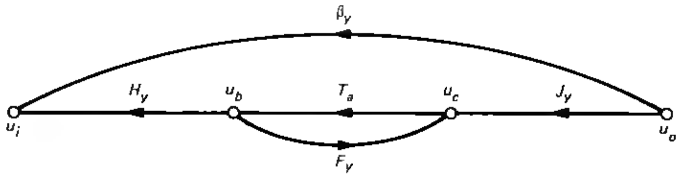
When active spanning networks are admitted, it is clear that any active, linear network can be represented as a ladder in the sense defined here. It also appears to be true, in looking ahead to the analysis of active spanning networks, tbat the direct, intuitive understanding of active circuits which comes from the elimination or gross reduction of return ratio can be substantially retained when active spanning networks are used.

VII. WRITING THE TRANSMISSION MATRIX EQUATION FROM THE EQUIVALENT LADDER CIRCUIT

When the feedback amplifier circuit has been redrawn in equivalent ladder form with spanning networks represented by dependent generators, a TMSFG can be drawn directly from the circuit by inspection. As a mechanical aide, a set of circuit vector nodes are placed on the circuit between each element of the ladder. These become the graph nodes of the TMSFG. Branches connecting these nodes in sequence from circuit output to circuit input (from graph input to graph output) define the



(a)



(b)

$$\begin{bmatrix} u_i \\ u_b \\ u_c \end{bmatrix} = \begin{bmatrix} H_y & 0 & \beta_y \\ 0 & T_a & 0 \\ F_y & 0 & J_y \end{bmatrix} \begin{bmatrix} u_b \\ u_c \\ u_o \end{bmatrix}$$

(c)

Fig. 16—(a) Equivalent ladder circuit, (b) TMSFG, and (c) transmission matrix array for a unitary C-feedback amplifier.

main transmission path. Each dependent generator will create a branch which spans one or more of these nodes: either a β branch in a direction from the circuit output toward the circuit input, or a direct feedthrough branch (an F branch) in a direction toward the circuit output. An example already examined in Section III is the C-feedback amplifier whose ladder circuit and TMSFG are shown in Fig. 16. The transmission matrix equation for the circuit is obtained as the transmission or graph gain of the TMSFG.

7.1 The transmission matrix array

Writing the transmission matrix equation of the amplifier is facilitated by putting the TMSFG into matrix form. Such a matrix form will be termed a *transmission matrix array* (TMA) which is itself a matrix relating the signal vectors at the nodes which receive signals to the signal vectors at the nodes which transmit them. The matrix elements are the branch values of the branches which connect these nodes in the TMSFG. In Fig. 16, for example, u_i, u_b, and u_c are nodes which receive signals; the signals at these nodes together form the *received signal vector*. Similarly, u_b, u_c, and u_o are nodes which transmit signals, which together form the *transmitted* signal vector. The matrix relating these two vectors

is the transmission matrix array, having a nonzero entry at any element position where a TMSFG branch transmits a signal from a component of the transmitted signal vector to a component of the received signal vector. Figure 16c shows the transmission matrix array for the circuit. It is evident that elements along the principle diagonal are matrices of the ladder network of cascaded circuit elements: elements above or to the right of the principle diagonal are β matrices, and elements below or to the left of the principle diagonal are F (direct feedthrough or feedforward) matrices.

Where all elements below the principle diagonal are zero or can be ignored, the transmission matrix equation relating u_i to u_o can be written by inspection. Thus, ignoring the direct feedthrough element in the TMA of Fig. 16, we can write

$$u_i \approx [\beta_y + H_y T_a J_y] u_o \tag{41}$$

To obtain the exact expression including direct feedthrough, the set of simultaneous equations represented by the TMA must be solved. Thus

$$u_c = F_y u_b + J_y u_o \tag{42}$$

$$= F_y T_a u_c + J_y u_o \tag{43}$$

so that

$$(I - F_y T_a) u_c = J_y u_o \tag{44}$$

and

$$u_c = (I - F_y T_a)^{-1} J_y u_o \tag{45}$$

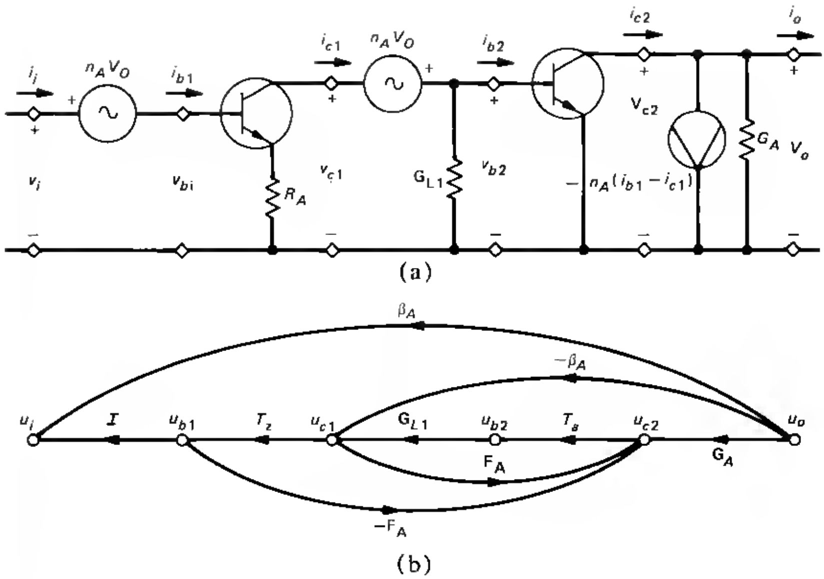
In this equation, J_y is premultiplied by the return difference matrix inverse, which, for small values of the elements of the return ratio matrix, $F_y T_a$, is essentially the identity matrix. In any case, substitution of $(I - F_y T_a)^{-1} J_y$ for J_y in the TMA of Fig. 16c removes the direct feedthrough element from the TMA, allowing us to write the transmission matrix equation by inspection:

$$u_i = [\beta_y + H_y T_a (I - F_y T_a)^{-1} J_y] u_o \tag{46}$$

The TMA can be written directly from the circuit diagram, allowing us to dispense with the TMSFG, its graph equivalent. In the more complicated feedback amplifiers to be discussed below, the TMA gives a clearer picture of signal dependencies than does the TMSFG.

7.2 Examples: feedback pairs

Figure 17a gives the equivalent ladder circuit for the A-feedback amplifier of Fig. 14. In Fig. 17b, we review the process of drawing the



$$\begin{bmatrix} u_i \\ u_{b1} \\ u_{c1} \\ u_{b2} \\ u_{c2} \end{bmatrix} = \begin{bmatrix} I & 0 & 0 & 0 & \beta_A \\ 0 & T_z & 0 & 0 & 0 \\ 0 & 0 & G_{L1} & 0 & -\beta_A \\ 0 & 0 & 0 & T_s & 0 \\ F_A & -F_A & 0 & 0 & G_A \end{bmatrix} \begin{bmatrix} u_{b1} \\ u_{c1} \\ u_{b2} \\ u_{c2} \\ u_o \end{bmatrix}$$

(c)

$$u_i \approx [\,\beta_A + T_z\,(-\beta_A + G_{L1}\,T_s G_A)]\,u_o$$

(d)

Fig. 17—The A-feedback pair. (a) Equivalent ladder circuit from Fig. 14d; (b) TMSFG; (c) TMA; (d) transmission matrix equation, ignoring direct feedthrough, written by inspection.

TMSFG. The input node, u_i, is identically the u_{b1} node except for the voltage-controlled voltage source, $n_A v_o$, so that u_i receives signals from two branches: the identity matrix branch from u_{b1} and the β_A branch from u_o. β_A is given by eq. (38). Next, u_{b1} is totally controlled by the transmission matrix of the transistor with R_A in the emitter, which we represent as T_z of Table I. Next, u_{c1} is equal to u_{b2} modified by the first-stage load conductance, G_{L1}, represented by the matrix

$$\mathbf{G}_{L1} = \begin{bmatrix} 1 & 0 \\ G_{L1} & 1 \end{bmatrix} \tag{47}$$

which appears as a TMSFG branch from u_{b2} to u_{c1}. In addition, the

voltage-controlled voltage source, $-n_A v_o$, adds branch $-\beta_A$ to u_{c1} from u_o. Node $u_{b2} = T_a u_{c2}$. Node u_{c2} has three inputs, $\mathbf{G}_A$ from u_o and the two F_A branches from u_{b1} and u_{c1} representing direct feedthrough of first-stage emitter current to the second collector. This completes the TMSFG.

The TMA can be constructed by exactly the same reasoning, and is shown in Fig. 17c. The columns of the TMA correspond to the transmitting nodes, and the rows to the receiving nodes. Where a node of a given row receives signal transmitted from a node of a given column, the branch transmission matrix is entered at the intersection, as shown.

The approximate transmission matrix equation, ignoring direct feedthrough, is written by inspection of the TMA as shown in Fig. 17d. This equation shows the overall input voltage augmentation by the first term, $\beta_A u_o$, and shows that the first stage incorporates local B-feedback, implicit in the transmission matrix, T_z. These are well-known characteristics of this feedback pair. What is less generally realized is that the second stage also incorporates local feedback, in this case local A-feedback, apparent from the additive $-\beta_A$ term in the parentheses, a term which is important to the high-frequency behavior of the circuit. (The input current from this cause alone is approximately $-C_1 n_A = n_A C_{cb1} s$.) The output loading of the feedback divider network is $G_A = 1/(R_E + R_F)$. If this is reduced by scaling R_E and R_F upward, the local feedback of the first stage is increased, since $R_A = R_E R_F/(R_E + R_F)$ is scaled up by the same factor, so that in the design of an A-feedback pair, a balance must be sought between these two effects.

The exact expression including the effect of direct feedthrough is useful as a final check, usually performed on the computer. It is obtained, as before, by solution of the simultaneous equations. We first reduce the TMA by direct substitution of the cascade portion of the TMA, that is, the portion containing no entries to the left of the principle diagonal. The TMA, thus condensed, is

$$\begin{bmatrix} u_i \\ u_{c2} \end{bmatrix} = \begin{bmatrix} T_z \mathbf{G}_{L1} T_a & (I - T_z)\beta_A \\ -F_A(I - T_z)\mathbf{G}_{L1} T_a & \mathbf{G}_A - F_A T_z \beta_A \end{bmatrix} \begin{bmatrix} u_{c2} \\ u_o \end{bmatrix} \tag{48}$$

Next, we remove the direct feedthrough term by removing the self-loop at node u_{c2}:

$$\begin{bmatrix} u_i \\ u_{c2} \end{bmatrix} = \begin{bmatrix} T_z \mathbf{G}_{L1} T_a & (I - T_z)\beta_A \\ 0 & M(\mathbf{G}_A - F_A T_z \beta_A) \end{bmatrix} \begin{bmatrix} u_{c2} \\ u_o \end{bmatrix} \tag{49}$$

Where M is the return difference matrix inverse, given by

$$M = [I + F_A(I - T_z)\mathbf{G}_{L1} T_a]^{-1} \tag{50}$$

The transmission matrix equation can now be written by inspection:

$$u_i = [(I - T_z)\beta_A + T_z \mathbf{G}_{L1} T_a M(\mathbf{G}_A - F_A T_z \beta_A)]u_o \qquad (51)$$

which reduces to the equation in Fig. 17d when $F_A = 0$.

The D-feedback pair whose equivalent ladder circuit is given in Fig. 15b can be analyzed by exactly similar means, with β_D given by eq. (39), G_D placed in shunt across the input terminals accounting for input circuit loading, and R_D in series with the emitter lead of the second transistor accounting for output loading by the spanning network. For this spanning network,

$$F_D = \begin{bmatrix} n_D & 0 \\ 0 & 0 \end{bmatrix} \qquad (52)$$

Writing the TMA and transmission matrix equation for this circuit is left as an exercise for the reader.

7.3 Hybrid feedback: the A/D hybrid feedback pair

Analysis of the A/D hybrid feedback pair demonstrates the utility of the transmission matrix array, as compared with the TMSFG. The TMA is essentially an incidence matrix of the TMSFG, presenting the same information in a better-ordered form. In Fig. 18a, the equivalent ladder circuit of Fig. 15d is repeated, and the TMSFG and TMA for this circuit are given in Fig. 18b and c. The TMSFG includes eight spanning branches; even if the four direct feedthrough branches are ignored, the tangle of β branches makes the writing of the graph gain (the transmission matrix of the circuit) hazardous. In the TMA, the role of each of these spanning branches is clarified, at least allowing us to write the approximate transmission matrix equation (in which the direct feedthrough branches are ignored) by inspection, proceeding row by row. Thus, the transmission matrix of the A/D feedback pair is written from the TMA as follows, in which the matrices G_D and G_A are first factored out, and the elements are considered row by row, starting from the right-hand end of the first row:

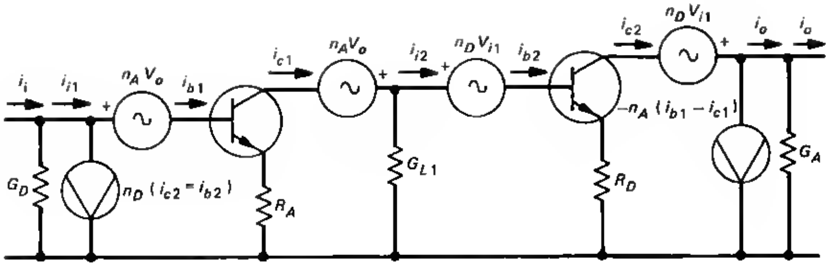
$$u_i \approx \mathbf{G}_d[\beta_A - \beta_D + \beta_D T_{z2} + T_{z1}(-\beta_A + \mathbf{G}_{L1} T_{z2})]\mathbf{G}_A u_o$$

or

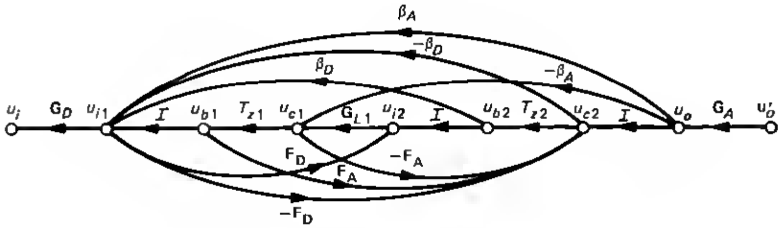
$$u_i \approx \mathbf{G}_D[(I - T_{z1})\beta_A - \beta_D(I - T_{z2}) + T_{z1}\mathbf{G}_{L1}T_{z2}]\mathbf{G}_A u_o \qquad (53)$$

The first term on the right in the brackets represents the A-feedback, the second term the D-feedback, and the third term the transmission matrix of the active path itself, modified by the series loading of the two spanning networks.

Equation (53) ignores the effects of the direct feedthrough branches, or the TMA entries below the principle diagonal, and is therefore approximate. The effect of the feedthrough branches is often to add excess



(a)



$$\beta_A = \begin{bmatrix} n_A & 0 \\ 0 & 0 \end{bmatrix} \quad \beta_D = \begin{bmatrix} 0 & 0 \\ 0 & -n_D \end{bmatrix} \quad F_A = \begin{bmatrix} 0 & 0 \\ 0 & -n_D \end{bmatrix} \quad F_D = \begin{bmatrix} n_D & 0 \\ 0 & 0 \end{bmatrix}$$

(b)

$$\begin{array}{c} \\ \begin{bmatrix} u_i \\ u_{i1} \\ u_{b1} \\ u_{c1} \\ u_{i2} \\ u_{b2} \\ u_{c2} \\ u_o \end{bmatrix} \end{array} = \begin{array}{c} \begin{array}{cccccccc} \underline{u_{i1}} & \underline{u_{b1}} & \underline{u_{c1}} & \underline{u_{i2}} & \underline{u_{b2}} & \underline{u_{c2}} & \underline{u_o} & \underline{u_o'} \end{array} \\ \begin{bmatrix} G_D & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & I & 0 & 0 & \beta_D & -\beta_D & \beta_A & 0 \\ 0 & 0 & T_{z1} & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & G_{L1} & 0 & 0 & -\beta_A & 0 \\ F_D & 0 & 0 & 0 & I & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & T_{z2} & 0 & 0 \\ -F_D & F_A & -F_A & 0 & 0 & 0 & I & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & G_A \end{bmatrix} \end{array} \begin{array}{c} \\ \begin{bmatrix} u_{i1} \\ u_{b1} \\ u_{c1} \\ u_{i2} \\ u_{b2} \\ u_{c2} \\ u_o \\ u_o' \end{bmatrix} \end{array}$$

(c)

Fig. 18—The equivalent ladder circuit, TMSFG, and TMA for the A/D hybrid feedback pair.

phase to the active path, and is therefore of importance in investigating stability in the vicinity of the crossover frequency (the frequency at which the magnitudes of the contributions to the loss ratio from the β path and the active path are equal). The complete transmission matrix, including the effect of direct feedthrough brances, is obtained by direct

but tedious algebra from the TMA:

$$T_{A/D} = (I - T_{z1})\beta_A - \beta_D M_2(W + I - F_D\beta_A) + (T_{z1}\mathbf{G}_{L1} + \beta_D)[M_1F_D(I - T_{z1})\beta_A + M_1T_{z2}M_2 (W + I - F_D\beta_A)] \quad (54)$$

where

$$M_1 = (I - F_DT_{z1}\mathbf{G}_{L1})^{-1}$$

$$M_2 = [I + [F_A(I - T_{z1}) - F_DT_{z1}]\mathbf{G}_{L1}M_1T_{z2}]^{-1}$$

and

$$W = [F_A(I - T_{z1}) + F_DT_{z1}][I - \mathbf{G}_{L1}M_1F_D(I - T_{z1})]\beta_A$$

A polynomial matrix manipulation computer program is of significant help in carrying out the indicated matrix operations. In such a program, currently in process of realization,* the elements of each of the matrices are put in a polynomial file, and the matrix operations indicated in eq. 54, for example, are carried out by simple commands. When the exact values of the transmission matrix elements have been found, the loss ratio and impedances can be found and automatically plotted.

While eq. (54) is far more complicated than (53), only two new matrices need be entered into file, namely F_D and F_A. Hence, the additional correction for the effects of direct feedthrough can be computed relatively easily, since most of the work involved in the computation is in entering the polynomial coefficients for the transistors and circuit elements into file. The instruction set for the computation consists essentially of the transmission matrix equation itself.

VIII. STABILITY

The above methods yield the transmission matrix of a feedback amplifier from which we derive a scalar measure of amplifier performance, such as loss ratio, in which we obtain the combined effect of the four matrix elements and the amplifier source and load immittances. For a linear, lumped-parameter circuit, the loss ratio will consist of a polynomial in the frequency variable, and may include a denominator polynomial, although this denominator often approximates unity. The condition for stability is that there shall be no roots of the (numerator) polynomial in the right-half plane of the complex frequency variable, since this would imply that, in the time domain, a growing exponential at the output could be supported with no input signal. The investigation of stability of distributed circuits, those containing transport delays, for example, is beyond our scope here, but these can be represented as

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lumped systems by use of polynomial approximants for delay, such as the Padé approximants.[30] Hence, in the transmission matrix approach to amplifiers, stability is ascertained by direct investigation of the properties of what the conventional approach calls the closed-loop gain, or, in the present analysis, its reciprocal.

There are two aspects to the study of stability: investigation of the stability of a given amplifier, and design of an amplifier to be stable. A more restrictive form of the latter is to require the amplifier to bave a prescribed transient response, since an amplifier which is merely stable may exhibit such damped oscillatory behavior as to be useless. The adjustment of the response of an amplifier to attain satisfactory transient response is termed *frequency compensation*, and involves adjustment of the coefficients of the loss ratio polynomial. In what follows, we shall study the case in which the loss ratio denominator is essentially unity.

Consider the loss ratio polynomial

$$L = \sum_{k=0}^{n} a_k s^k \tag{55}$$

We begin by normalizing the polynomial, to make the first and last terms unity, first by dividing throughout by a_0:

$$L = a_0 \left(1 + \sum_{k=1}^{n} \frac{a_k}{a_0} s^k\right) \tag{56}$$

Next, we change the frequency variable so that the coefficient of the highest-order term in the brackets is unity by letting

$$\frac{a_n}{a_0} s^n = p^n$$

or

$$s^k = \left(\frac{a_0}{a_n}\right)^{k/n} p^k \tag{57}$$

Thus,

$$L = a_0 \left(1 + \sum_{k=1}^{n-1} \frac{a_k}{a_n{}^{k/n}\, a_0{}^{1-k/n}} p^k + p^n\right) \tag{58}$$

The loss ratio is now in the desired form for investigation of stability and transient response. It may be written

$$L = a_0(1 + b_1 p + b_2 p^2 + \ldots + b_{n-1} p^{n-1} + p^n) \tag{59}$$

All information about the stability and transient response is contained in the values of the coefficients b_1 to b_{n-1}. In a cubic polynomial, for example, two coefficients, b_1 and b_2, determine the transient response.

Table III — Pascal-like triangles for normalized coefficients of the loss ratio polynomial

n	Multiple pole	Butterworth	Thomson
0	1	1	1
1	1 1	1 1	1 1
2	1 2 1	1 1.41 1	1 1.73 1
3	1 3 3 1	1 2 2 1	1 2.47 2.43 1
4	1 4 6 4 1	1 2.61 3.41 2.61 1	1 3.20 4.39 3.12 1

For a given set of polynomial coefficients, the roots are investigated to see if any lie in the right-half plane, in which case the amplifier is unstable. (For this determination, the normalization is unnecessary.) This is the only stability criterion necessary, since the design focuses on the performance of the amplifier, not a feedback loop. In traditional analysis, the focus was on the feedback loop and its analysis and design, so that an additional step, that of relating the closed-loop performance to the loop gain, had to be taken. The Nyquist criterion and its many later reinterpretations were worked out to ease this step. In the present method, these rather elaborate procedures are unnecessary. The loss ratio is found as the sum of the active-path and β-path contributions, and since the active path is usually expressible as a polynomial rather than a ratio of polynomials, the addition is simply made by adding the polynomial coefficients of the two paths. Denominators do arise. In the active path, these come from direct feedthrough and sometimes from frequency compensation networks which are used to adjust the polynomial coefficients to secure a prescribed transient response. In the β path, denominators arise when this path is used for equalization and filter applications. In these cases, we have no choice but to do the necessary multiplications to put both path polynomials over a common denominator.

In amplifier design where the denominators are incidental, prescribed transient response is obtained by designing the circuit such that the b coefficients of eq. (59) satisfy the performance criteria. Conversely, we may take the b coefficients as a performance specification for the amplifier. Examples of such criteria are given in Table III, which lists the b-values for an amplifier having either Butterworth or Thomson response characteristics.[31] Circuit methods for the adjustment of the b values to agree with a set of values such as those of Table III are beyond our scope in this paper, but a few comments are in order. The value of a_0 in eq. (59) is primarily established by the β path where the benefits of feedback (input augmentation) are to be obtained. The original reason for this was to reduce distortion introduced by the active devices, since the β path is linear and the active path is not, so that the β-path contribution was arranged to swamp out the smaller nonlinear contribution to the input signal. The second coefficient, a_0b_1, as well as the remaining

frequency-dependent coefficients are ordinarily supplied by the active path, although it may be advantageous to have the β path supply and even dominate a_0b_1. In the case of C-feedback, for example, the a_0b_1 term is augmented by connecting a capacitor (a linear capacitor) between the output and the input in parallel with the β-path conductance. The a_0b_2 coefficient is adjusted upward by connecting capacitive feedback internal to the active path such that the capacitive current thus generated is multiplied by one of the active device matrix elements which is proportional to frequency, thereby augmenting the second-order coefficient, and so on through the set of b coefficients.

IX. DISCUSSION AND CONCLUSIONS

Viewed from both practical and theoretical standpoints, the process of analyzing, designing, and even thinking about active two-port circuits is simplified by taking an anticausal approach to the functional dependencies in the circuit. It does this because the importance of feedback or loop gain is greatly reduced, and with it denominators of the circuit expressions, which no longer depart greatly from unity.

The specific method described here for anticausal analysis of circuits is to base their transducer characteristics on the transmission matrix. This matrix puts cascades of two-ports into anticausal form directly, leaving the problem of how more remote circuit coupling is to be accommodated. In the method described here, such coupling is taken to be the property of spanning networks, which are described by the appropriate set of two-port parameters (h, z, y, or g). Each such spanning network parameter set yields four separate transmission matrices, each containing one of the four spanning network parameters, and each corresponding to one of four effects which are to be accounted for when the spanning network is applied to modify the amplifier characteristics. The 11 parameter and its associated transmission matrix corresponds to input circuit loading by the feedback network; the 12 parameter and its associated β matrix represents the input signal augmentation corresponding to the feedback signal of conventional analysis; the 21 parameter yields a transmission matrix which accounts for direct feedthrough of signals from circuit input to output through the spanning network; and the 22 parameter represents output circuit loading by the spanning network.

With all circuit element characteristics expressed as transmission matrices, it is desirable to be able to describe the whole circuit in these terms. The transmission matrix signal-flow graph, with its one-to-one correspondence between circuit vector nodes and graph nodes, provides a means for writing the transmission matrix equation of the whole circuit from the individual transmission matrices and their topological relationships. The transmission matrix array is a clearer way of showing the functional dependencies established by the transmission matrix sig-

nal-flow graph. From either of these two artifices, the transmission matrix equation for the whole may be written directly. This transmission matrix equation can be used for an initial look at a circuit to establish the basic properties of the configuration, by making suitable approximations such as that obtained by placing the transistors in their reference condition, through more accurate intermediate levels of approximation, by including the more important transistor parameters. Finally, an exact transmission matrix for the whole circuit may be derived, within the accuracy of the transistor and circuit element characterization available, traceable from initial approximation to final result.

Many problems remain, the most immediate of which is to complete the computational tools for linear analysis, and beyond that, the extension to quasilinear analysis and distortion, for which the present approach appears to offer substantial benefits. Active device characterization should be done in terms of anticausal functional dependencies: for linear analysis, we require transmission parameters of the active devices, an example of which is shown in Fig. 5. Nonlinear characterization of the partial derivatives of eq. (3) expressing the input signal vector as a function of the output, is needed. The noise of a two-port can be expressed as an equivalent input noise network including a series voltage generator and a shunt current generator.[32] It should be convenient to express not only the noise, but the predistortion, the dc input offsets, and the variation of the input signal vector due to transistor parameter variations, as an "input uncertainty network" consisting of a series voltage generator and a shunt current generator which in sum include all of these effects.

Beyond the two-port analysis discussed here, there are many instances where multiport analysis is needed. As a simple example, the operational amplifier with its positive and negative input leads can be considered a three-port (leaving out the power supply leads, which in most applications are at signal ground). The circuit partitioning resulting from the separation between the device supplier and user requires a three-port characterization, and with it, resolution of the question of functional dependencies of the six signal variables involved, comprising the two input signal vectors and the output signal vector.

It may be time to rid ourselves of the notion of feedback as a central concept in analysis of electronic amplifiers and other deterministic physical systems. As it applies to mercantile or social systems, where the reaction to a given event is barely predictable, the idea may still be of use, as for example, in the Club of Rome report.[33] Even in this area, anticausal analysis may supplant it. In project management, the PERT system, originally applied to the Polaris submarine, starts with the project goal and its projected date of completion, and works back to distinct events which must have happened to reach the goal.[34] In a sense,

the PERT chart is a TMSFG, without feedback loops. In comparison, the flow diagram of the world model (Fig. 26 of Ref. 33) is a feedback-loop-filled diagram inaccessible to human understanding. Were the projected goals introduced in that report used as flow graph *inputs,* the complex interrelationships among the variables might have been more readily understood. The mathematical description of feedback came out of the development of electronic amplifiers for carrier transmission, and has been widely adopted in other areas. The alternative suggested here might also find use in other areas.

X. ACKNOWLEDGMENTS

The work reported here is enough different from conventional amplifier theory that the author has had to rely on the experience and insights of many people. For the initial impulse to formalize an amplifier design method which the author has been using for some time, he is indebted to William McGee of Bell Northern Research, who invited him to give a paper in Toronto in 1973. For the kind of encouragement that only practical experimental realization can give, the author is indebted to Dan Wolaver and Walter Kruppa for their work on a UHF integrated operational amplifier, which was designed by use of the methods reported here. Special thanks are due Abraham Osofsky for developing a computer program (to be described in a future publication) which greatly enhances the usefulness of the method, allowing us to complete the traceable path from approximation to exact characterization.

The formalization of the method was advanced significantly by many class discussions during an in-hours course on amplifier design taught by the author in the fall of 1975; the efforts of the students of this course allowed the author to avoid many hazards of fuzzy thinking. For valuable discussions arising from this course, the author is indebted to J. A. Bellisio, W. I. H. Chen, D. L. Duttweiler, S. D. Personick, and J. M. Sipress. J. C. Candy and L. A. O'Neill offered valuable advice on the manuscript. Finally, the constant encouragement and insights of M. R. Aaron, who reviewed the key concepts as they evolved, are deeply appreciated.

GLOSSARY

Several terms have been introduced in this paper. For convenience, they are gathered here with brief definitions and the section number where they first appear.

Transmission matrix signal flow graph (III). A signal flow graph having signal vectors at circuit vector nodes for graph nodes and transmission matrices for branches.

Vector node or circuit vector node (III). A circuit node having only two

connections to it, allowing us to define uniquely the node voltage (to ground) and the node current, which together form the signal vector at the corresponding TMSFG node.

Loss ratio (III). The ratio of the generator voltage, e_G, to the output voltage, v_o, of a two-port circuit connected between a Thevenin source, e_G, R_G, and a load conductance, G_L.

Cascade graph (III). A signal flow graph or TMSFG having no feedback loops.

Spanning network (IV). A two-port network connected between two nonadjacent pairs of circuit vector nodes of a cascaded or ladder network. A spanning network is represented by one of the four sets of two-port parameters, h, z, y, or g.

Input signal augmentation or feedback (IV). The increase in input signal voltage or current (at constant output) due to the action of the spanning network; in particular, input augmentation is due to the 12 parameter (such as y_{12}) of the spanning network.

Direct feedthrough or *feedforward* (IV). The increase of change in the output signal voltage or current due to the action of the spanning network; in particular, direct feedthrough is due to the 21 parameter (such as y_{21}) of the spanning network.

Input circuit loading (IV). Shunt or series loading of a ladder network by the 11 parameter (such as y_{11} or z_{11}) of the spanning network.

Output circuit loading (IV). Shunt or series loading of a ladder network by the 22 parameter (such as y_{22} or z_{22}) of a spanning network.

β matrix (IV). A transmission matrix containing one nonzero element equal to the 12 parameter of a spanning network. Usually carries a subscript indicating which parameter set it is associated with, as in eq. (7).

F matrix (IV). A transmission matrix containing one nonzero element equal to the 21 element of the spanning network. Also called the direct feedthrough matrix. See eq. (8).

H matrix or *input loading matrix* (IV). A transmission matrix which is the sum of the identity matrix and a matrix having one nonzero element equal to the 11 parameter of the spanning network, as in eq. (9).

J matrix or *output loading matrix* (IV). A transmission matrix which is the sum of the identity matrix and a matrix having one nonzero element equal to the 22 parameter of the spanning network, as in eq. (10).

Return ratio matrix (IV). A transmission matrix equal to the loop gain of a feedback loop in a TMSFG.

Return difference matrix inverse (IV). A transmission matrix which postmultiplies the active path transmission matrix to account for the effect of a feedback loop.

Reference condition for a feedback circuit (V). A feedback circuit in which the active element(s) is (are) replaced by ideal two-port amplifier(s).

Ideal two-port amplifier (V). An amplifier whose transmission matrix is null.

Unitary feedback amplifier (V). An amplifier whose transmission matrix contains but one nonzero element when its active elements are placed in the reference condition.

Hybrid feedback amplifier (V). An amplifier whose transmission matrix contains more than one nonzero element when its active elements are placed in the reference condition.

Equivalent ladder circuit (VI). An equivalent circuit, drawn in ladder form, with remote couplings expressed by dependent generators.

Transmission matrix array (VII). An incidence matrix of the TMSFG which relates the received signals at a set of nodes to the transmitted signals at a set of nodes.

Received signal vector (VII). The set of signals at all nodes of a TMSFG which receive signals.

Transmitted signal vector (VII). The set of signals at all nodes of a TMSFG which transmit signals.

APPENDIX A

In what follows, we solve the simultaneous equations for the C-feedback amplifier of Fig. 6a. The TMSFG of Fig. 8b is repeated in Fig. 19a from which we can write (leaving out loading matrices H_y and J_y for the moment)

$$u_i = \beta_y u_o + u_b \tag{60}$$

$$u_b = T_a u_c \tag{61}$$

$$u_c = F_y u_b + u_o \tag{62}$$

Substituting (61) in (63), we form a self-loop at node u_c, as shown in Fig. 19b:

$$u_c = F_y T_a u_c + u_o \tag{63}$$

in which the matrix $F_y T_a$ will be called, following Bode's notation, the *return ratio matrix*. Solving for u_c, we have

$$u_c = (I - F_y T_a)^{-1} u_o \tag{64}$$

The matrix $I - F_y T_a$ corresponds to Bode's return difference, so that we term $(I - F_y T_a)^{-1}$ the *return difference matrix inverse*. Substituting (64) in (61), and thence in (60), we obtain

$$u_i = [\beta_y + T_a (I - F_y T_a)^{-1}] u_o \tag{65}$$

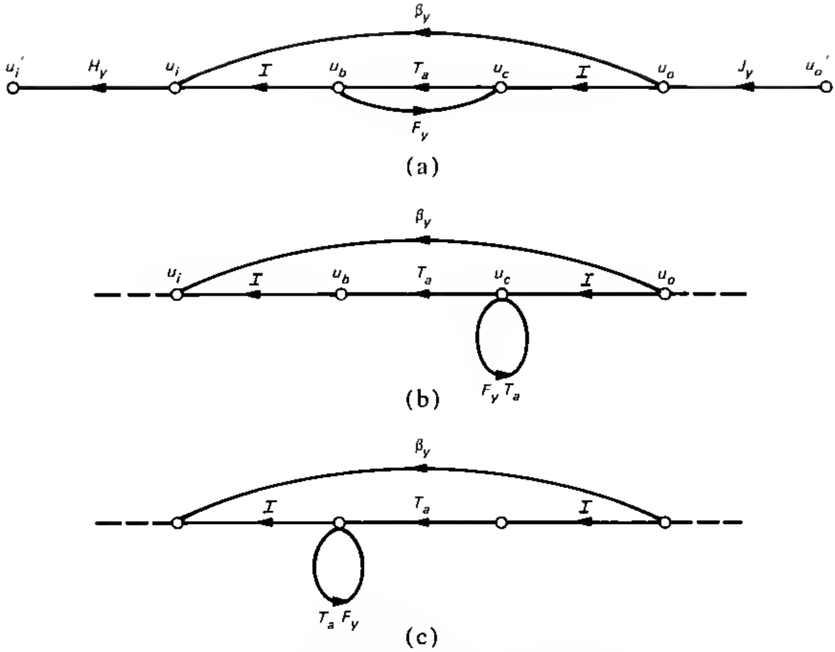

Fig. 19—Alternative TMSFG reductions of the C-feedback amplifier. (a) TMSFG from Fig. 8b; (b) creation of a self-loop at node u_c; (c) creation of a self-loop at node u_b. Tbe preferred form is that of (b).

The transmission matrix for the stage is obtained by premultiplying by the input loading matrix, H_y, and postmultiplying by J_y:

$$T_y = H_y[\beta_y + T_a(I - F_yT_a)^{-1}]J_y \tag{66}$$

which is eq. (11) of the text.

Alternatively, we could solve the simultaneous equations by substituting (62) in (61), forming a self-loop at node u_b, as shown in Fig. 19c:

$$u_b = T_aF_yu_b + T_au_o \tag{67}$$

where T_aF_y is the new return ratio matrix. Solving for u_b, we also obtain a new return difference matrix inverse:

$$u_b = (I - T_aF_y)^{-1}T_au_o \tag{68}$$

from which we obtain

$$u_i = [\beta_y + (I - T_aF_y)^{-1}T_a]u_o \tag{69}$$

Clearly, $T_aF_y \neq F_yT_a$, since, as anyone knows who has tried to clean his glasses and blow his nose with the same tissue, the order in which the operation is carried out is important. It should not disturb the reader

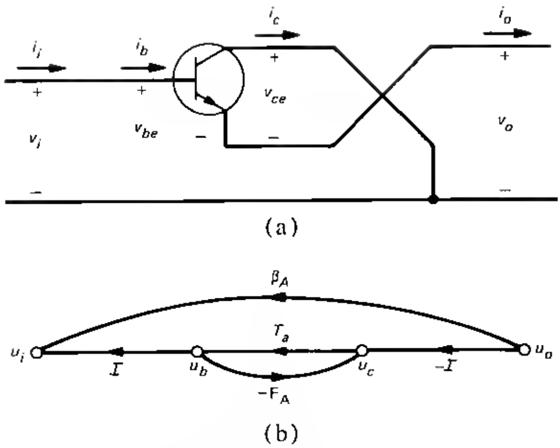

Fig. 20—Common collector stage. (a) Circuit and (b) TMSFG representing the transmission matrix equation derived in the text.

that the return ratio and return difference are dependent upon the way in which we solve the simultaneous equations, since these quantities are not invariants of the circuit, but depend entirely upon how we view the circuit.[15] On the other hand, comparing eqs. (65) and (69), $T_a(I - F_yT_a)^{-1} = (I - T_aF_y)^{-1}T_a$, so that the transmission matrix of the active path is invariant. For linear analysis, we are free to use either formulation. When we consider the extension to nonlinear analysis, however, eq (65) is preferred, since it preserves the actual signal level at u_c, the output node of the active device.* We therefore adopt a rule of procedure for solving simultaneous circuit equations: Always preserve the output node. This is done by placing the self loop at the output of a device exhibiting direct feedthrough, and allows a straightforward calculation of the waveform at the output of a nonlinear device. As a matter of practice, the node equation for a node nearer the input should be substituted into the node equation for a node nearer the output.

APPENDIX B

Common collector stage

The common collector stage is shown in Figure 20a. The circuit equations are written starting at the input:

$$v_i = v_b + v_o \tag{70}$$

$$i_i = i_b$$

* The advantage of the formulation of eq. (65) over that of eq. (69) was pointed out to the author by C. A. Desoer.

or

$$u_i = u_b + \beta_A u_o \tag{71}$$

where

$$\beta_A = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} \tag{72}$$

Next,

$$u_b = T_a u_c \tag{73}$$

and

$$\begin{aligned} v_{ce} &= -v_o \\ i_c &= -i_o + i_b \end{aligned} \tag{74}$$

or

$$u_c = -u_o - F_A u_b \tag{75}$$

where

$$F_A = \begin{bmatrix} 0 & 0 \\ 0 & -1 \end{bmatrix} \tag{76}$$

Substituting (73) into (75), and solving for u_c, we have

$$u_c = -(I + F_A T_a)^{-1} u_o \tag{77}$$

By successive substitution, we obtain the input vector as a function of the output:

$$u_i = [\beta_A - T_a(I + F_A T_a)^{-1}] u_o \tag{78}$$

To evaluate the matrix of the common collector stage, we obtain the return ratio matrix:

$$F_A T_a = -\begin{bmatrix} 0 & 0 \\ C & D \end{bmatrix} \tag{79}$$

The return difference matrix inverse is

$$(I + F_A T_a)^{-1} = \frac{1}{1-D} \begin{bmatrix} 1-D & 0 \\ C & 1 \end{bmatrix} \tag{80}$$

and the matrix for the stage is

$$T_{cc} = \frac{1}{1-D} \begin{bmatrix} 1 - A - D + \Delta^t & -B \\ -C & -D \end{bmatrix} \tag{81}$$

as given in eq. (33). From eq. (78), we can draw a TMSFG for the stage as shown in Fig. 20b.

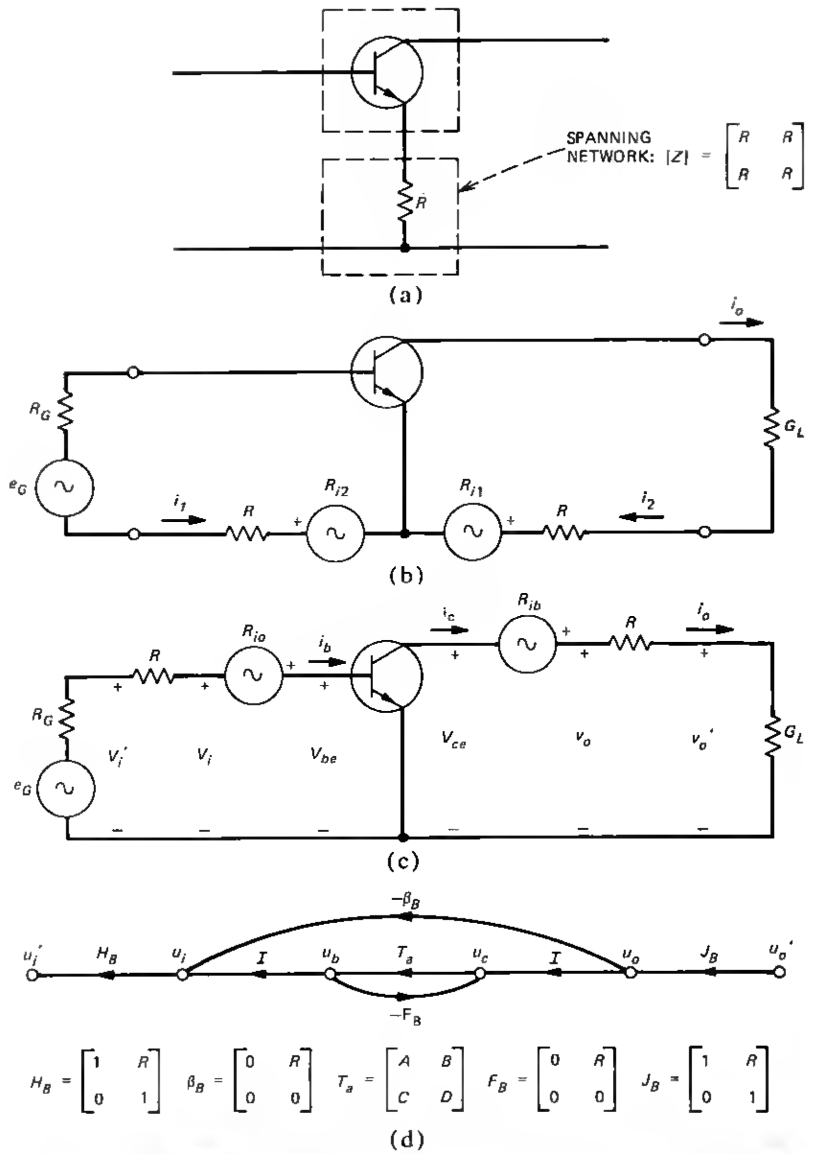

Fig. 21—Emitter resistor feedback analysis. (a) Circuit. (b) Circuit redrawn using dependent generator equivalent circuit of Fig. 7. (c) Redrawn circuit, interchanging position of series elements. (d) TMSFG witb definitions of the matrices.

APPENDIX C

Emitter resistor feedback

A transistor with unitary B feedback is shown in Fig. 21. In (a), the circuit is divided into an active path and a resistive spanning network,

and in (b), the spanning network is represented by its dependent generator equivalent circuit from Fig. 7b. A common ground does not exist between the input and output loops, but the circuit as drawn in (b) is nevertheless a two-port, since $i_1 = -i_b$, and $i_2 = i_o$ with the current directions given in the figure. An equivalent representation is given in (c) of the figure, in which elements in series have had their circuit positions interchanged. The TMSFG for the circuit of (c) is given in (d), in which H_B and J_B represent the series input and output resistances, $-\beta_B$ represents the generator in series with the input lead, and F_B is the direct feedthrough supplied by the generator in series with the output lead. From the TMSFG, the transmission matrix equation is

$$T_z = H_B[-\beta_B + T_a(I + F_B T_a)^{-1}]J_B \tag{82}$$

With the element values of the matrices given in the figure, the return ratio matrix is

$$F_B T_a = \begin{bmatrix} CR & DR \\ 0 & 0 \end{bmatrix} \tag{83}$$

and the return difference matrix inverse is

$$(I + F_B T_a)^{-1} = \frac{1}{1 + CR}\begin{bmatrix} 1 & -DR \\ 0 & 1 + CR \end{bmatrix} \tag{84}$$

from which the active path matrix without loading becomes

$$T_a(I + F_B T_a)^{-1} = \frac{1}{1 + CR}\begin{bmatrix} A & B - R\Delta^t \\ C & D \end{bmatrix} \tag{85}$$

Adding the input augmentation from $z_{12} = R$, we have

$$-\beta_B + T_a(I + F_B T_a)^{-1} = \frac{1}{1 + CR}\begin{bmatrix} A & -R(1 + CR) + B - R\Delta^t \\ C & D \end{bmatrix} \tag{86}$$

Finally, we premultiply by H_B and postmultiply by J_B, and obtain

$$T_z = \frac{1}{1 + CR}\begin{bmatrix} A + CR & B - R(1 - A - D + \Delta^t) \\ C & D + CR \end{bmatrix} \tag{87}$$

as shown in eq. (35).

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